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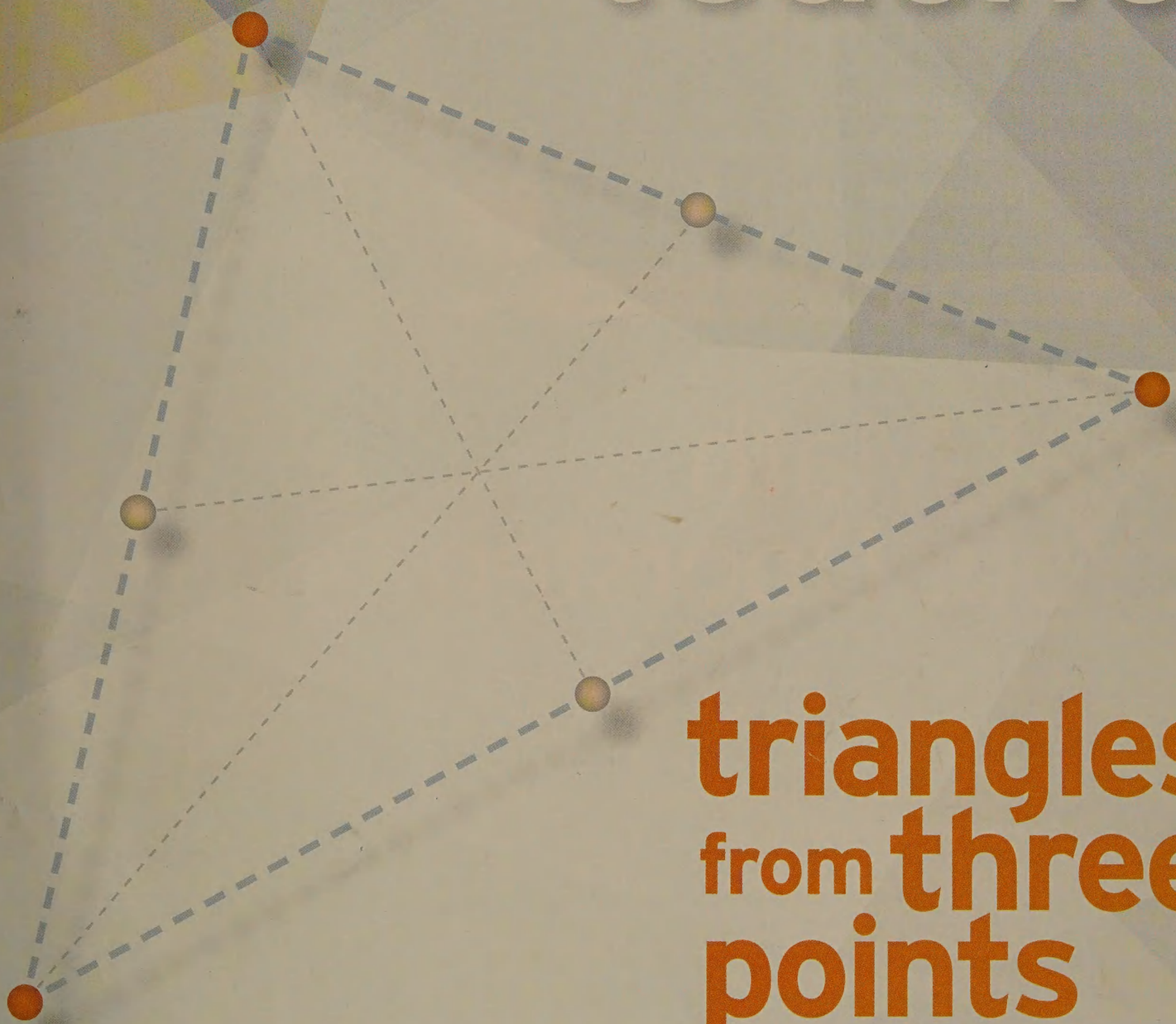
AUGUST 2014



triangles from three points

Who's in the Lead?
Winning with the
Law of Large Numbers

Quantitative **Reasoning**
and Mathematical **Modeling**

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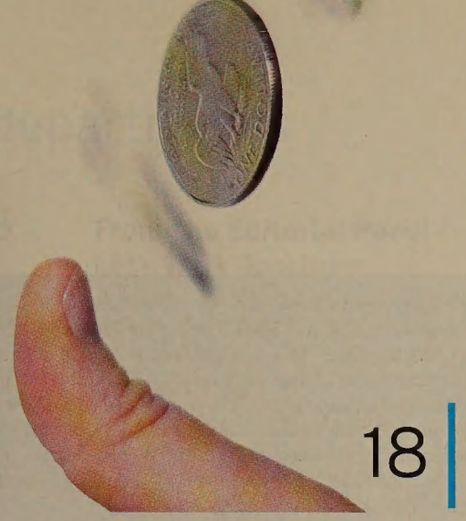

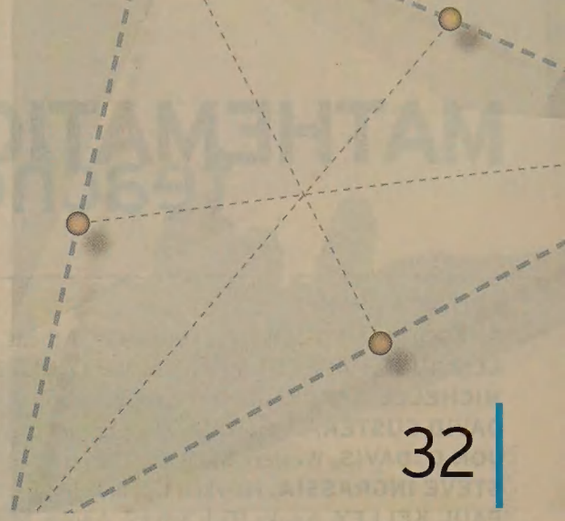

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Not all problems are well suited as projects, but Wayne Nirode has used this gem—"Triangles from Three Points" (pp. 32–38)—with his honors geometry class. Consider how you and your students might investigate this problem with learning opportunities to reinforce algebra skills and geometry concepts or explore dynamic constructions and explain mathematical reasoning. Coordinate geometry at its best!

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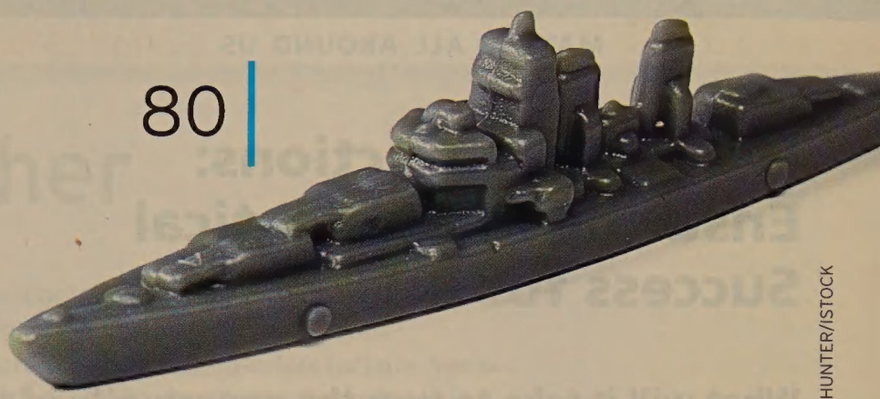

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- "Activities for Students" (p. 58): Customizable activity sheets
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in the next issue

Coming in September 2014:

- "Probability and Perception: The Representativeness Heuristic in Action," by Yun Lu, Francis J. Vasko, Trevor J. Drummond, and Lisa E. Vasko
- "Pose and Solve Varignon Converse Problems," by José N. Contreras
- "Implementing the Common Core: Applying Shifts to Instruction," by Angeline K. Gaddy, Shannon E. Harmon, Angela T. Barlow, Charles D. Milligan, and Rongjin Huang

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Let's Work Together

In recent years, schools and teachers have been placed under the microscope and even under attack. Evidence such as the great need for college mathematics courses reviewing preparatory content makes it clear that mathematics learning in North America is far from perfect. The question is, How should we proceed?

To improve school mathematics, we must work together as a community and share our best ideas and resources to support student learning. Let's stand arm in arm as we face ongoing challenges in the classroom. Working together as professionals has been central to the mission of the National Council of Teachers of Mathematics since it was founded nearly a century ago. And, even longer, *Mathematics Teacher* has served as a vehicle for teaching professionals to disseminate information, investigate concepts, and discuss pedagogy.

As another school year begins, *Mathematics Teacher* is here to help us bring the subject that we love to life for our students. The Editorial Panel urges you to submit manuscripts about how you engage your students in mathematical tasks that develop knowledge and skills; that promote quantitative, visual, and logical thinking; and that foster communication.

We need your ideas about the traditional topics of number, algebra, functions, and geometry and the increasingly important topics of quantitative reasoning, modeling, discrete mathematics, probability, and statistics. How do you use technology in meaningful ways to help students access difficult concepts and explore rich applications?

Current calls for manuscripts include Assessment; Great Problems; On the Front Burner: Emerging Issues in Mathematics Education; and the 2016 Focus Issue: Teaching Mathematics Online. Please consider contributing to one of these high-priority topics.

The *MT* departments—Reader Reflections, Sound Off!, Media Clips, Mathematical Lens, Calendar, Activities for Students, Connecting Research to Teaching, Technology Tips, Delving Deeper, For Your Information, and The Back Page—are designed to serve the diverse needs and interests of our readers. Departments provide an excellent opportunity for new authors to contribute their best ideas to the national conversation about enriching school mathematics. One change for this volume year is that Problem Solvers has been retired (see p. 45).

The theme of the 2014 *MT* Focus Issue is What Makes Ideas Stick? This timely topic will be spotlighted in the November issue, and other articles throughout the volume year will address how to make mathematics memorable.

As the new school year begins, the Editorial Panel offers you best wishes for a productive and rewarding year, and we hope that you will consider sharing your best ideas with your colleagues through *MT*. By working together we can "ensure equitable mathematics learning of the highest quality for all students."

The Editorial Panel of *Mathematics Teacher*: (back row) Paul Kelley, David Custer, Jon Davis, Megan Snow, Greg Foley; (front row) Elizabeth Applebaum, Laurie Rubel, Tara Slesar, Michelle Berry, Alison Langsdorf, Steve Ingrassia

READER reflections

A CALCULATOR'S LIMITS

While I was teaching a precalculus class, one question asked us to find the intersection for the graphs of $f(x) = x^3 - x$ and $g(x) = \sqrt{x+1}$. The function $f(x)$ has zeros at $x = -1$, 0, and 1, and $g(x)$ has a zero at –1, so one solution is (–1, 0). The other solution cannot be found algebraically but can be approximated to be at $(x, y) \approx (1.4432, 1.5631)$ using the **Intersect** option.

The interesting thing is that the calculator will not find the intersection at (–1, 0). If, using the Newton method for approximations of zeros and intersections, we encounter a stationary point, the derivative is zero and the method will terminate because it would require division by zero. This is an interesting way for students to learn that a calculator does have limitations.

Roger L. Guffey
rlguffey1@twc.com
Bluegrass Community and Technical College and the University of Kentucky
Lexington, KY, Sept. 30, 2013

AN OLD FORMULA REVISITED

Calculus students often have difficulties setting up the integrals to find the volume when a region bounded by $R(x)$ and $r(x)$, or y_{upper} and y_{lower}, is rotated around some horizontal line. It seemed that there might be a more general formula to use with the disk method, so we set out to find one.

Tips can be offered, such as considering whether or not the representative rectangle "touches" the axis of rotation, but students still find themselves in the position of deciding "Do I subtract and then square, or do I square and then subtract?" That is, should we use

$$\left(R(x)-r(x)\right)^2 \text{ or } \left(R(x)\right)^2-\left(r(x)\right)^2?$$

The first expression leads to a disk if $r(x)$ is the same as the axis of rotation, whereas the second expression leads to a washer.

After working through several problems, we found that one formula could be used for both cases:

$$\text{Volume} = \pi\int_a^b\left(\left(y_{upper}-y_{axis}\right)^2-\left(y_{lower}-y_{axis}\right)^2\right)dx$$

When the lower boundary is the axis of rotation, we have the first expression. When the axis of rotation is the x-axis, $y_{axis} = 0$, we recover the second expression above. (Unfortunately, this formula has found its way into the extra-hard problems of calculus texts [see Briggs and Cochran 2011] but not into the main body of the text.)

The point of this reflection is to encourage other teachers to try this formula. You will see that students can set up the integrals much faster.

Lance Hemlow
lhemlow@raritanval.edu
Raritan Valley Community College
Somerville, NJ, Oct. 1, 2013

SECOND FRACTION POINTS

When I was a teenager (I am 81 now), my Uncle Ed challenged me, asking me what a second fraction point, such as that in 12.34.5, would mean. In a few minutes, I saw that 7.3 meant 7 + 1/10 of 3 and that 9.56 meant 9.5 + 1/100 of 6, or 9.5 + .06.

It followed that .34.5 would mean .34 + 1/1000 of 5, or simply .345, so the second point was redundant. It means *nothing at all!* (It is similar to writing .3 as 0.3 or writing 0 as 00.)

Thus, 0.2½ is 0.2.5. By deleting the redundant second point, we obtain 0.25, or 1/4. Likewise, since 5/11 = .454545..., .45$\frac{5}{11}$ = .45.454545..., or simply .45454545..., which in turn is 5/11.

Gene Zirkel
genezirk@gmail.com
Professor emeritus
Nassau Community College
Garden City, NY, Nov. 9, 2013

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

MORE ON *a* QUADRATIC FUNCTION

I very much enjoyed reading Jon D. Davis's article "An Unexpected Influence on *a* Quadratic" (*MT* Oct. 2013, vol. 107, no. 3, pp. 212–18). In it he discussed the path the vertex of a parabola takes when the value of a in the function $y = ax^2 + bx + c$ is changed along a continuum. He demonstrated that this path is a line with the equation $y = (b/2)x + c$.

My precalculus classes were studying transformations, and I felt that this would be a good time to discuss the article with them. Using The Geometer's Sketchpad®, the students graphed a quadratic function with specific numerical values for b and c and set up the a value as a parameter. They then located the vertex using **Advanced Toolkit**, animated the a parameter, and used **TRACE** to follow the vertex point. Indeed, a line was traced. The students correctly surmised that the parameters in the equation of this line should be borrowed from the quadratic. We came to the same conclusion that Davis did.

My students wanted to continue investigating graphs of quadratics and decided to trace the vertex again, this time making the b coefficient the parameter. We used the same procedure that we used in animating the a parameter and got another unexpected result. To our surprise, the path traced was a parabola! An image of this graph is shown in **figure 1 (Schuloff)**.

Of course, we wanted to find the parabola's equation. The class noticed that the vertex was the same point as the y-intercept of our original parabola and that the new parabola was symmetrical to the y-axis. Figuring out the equation did not seem to be a formidable task. Of course, our a value needed to be negative, and we decided to use the a value from our graph, which, in this case, was 0.5. Lo and behold, our equation of $y = -0.5x^2 - 3$ perfectly overlaid the trace! See **figure 2 (Schuloff)**.

We concluded that if the b value of a quadratic function is changed, then the vertex (and the entire graph) travels in a parabolic path along $y = -ax^2 + c$.

Anita Schuloff
aschuloff@paramus-catholic.org
Paramus Catholic High School
Paramus, NJ, Nov. 12, 2013

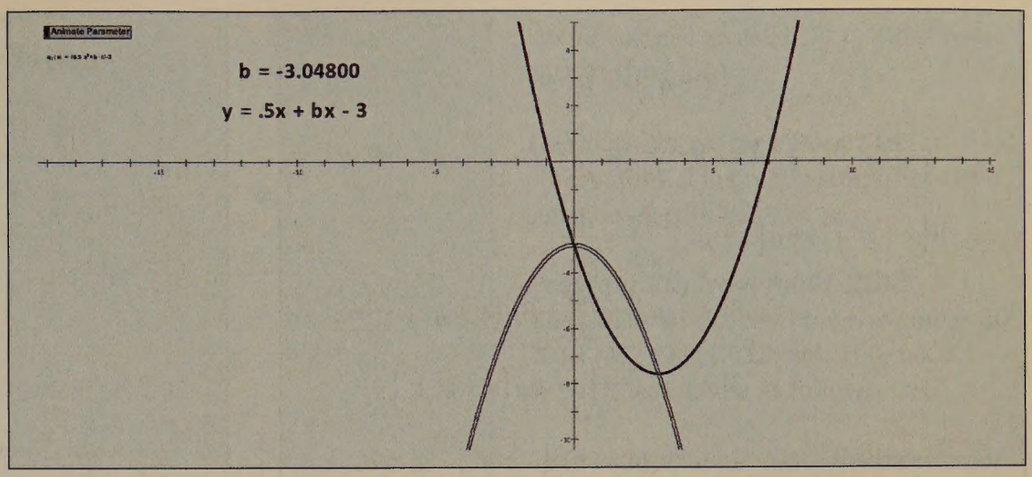


Fig. 1 (Schuloff)

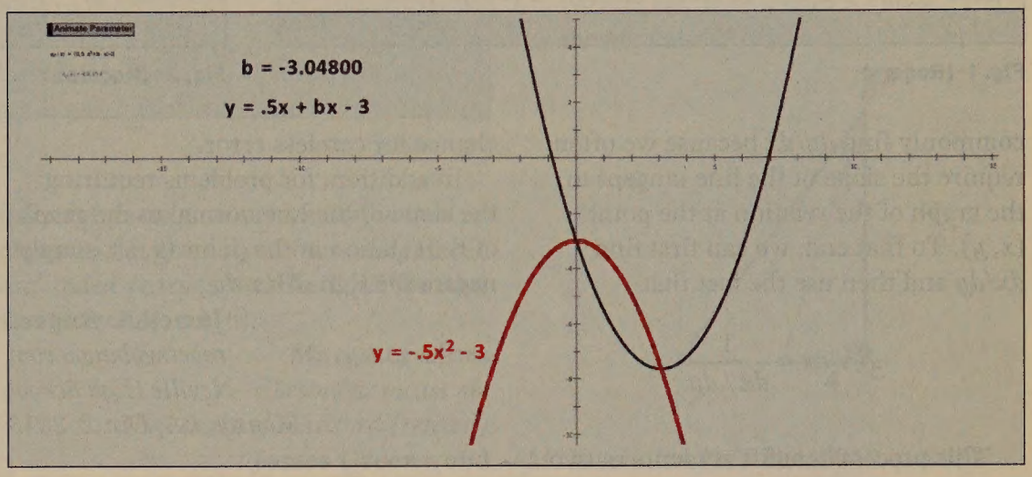


Fig. 2 (Schuloff)

Editor's note: For another look at parameters in the equations of conic sections, including ellipses and hyperbolas, see "Technology-Enhanced Discovery," by Chris Harrow and Lillian Chin, *MT* May 2014, vol. 107, no. 9, pp. 660–65.

PERFECT SQUARE BASES

In Richard Forringer's reflection on problems 22 and 23 from the May 2012 Calendar (*MT* Sept. 2013, vol. 107, no. 2, p. 85–86), he asks for the next three bases in the sequence after 176 that will express 31 in those bases as a perfect square. The next three bases in the sequence after 176 are 208, 225, and 261. Note that $31_{208} = 625_{10} = (25_{10})^2 = (P_{208})^2$, where P is used to represent 25 in base 208.

A simple formula can be used to find certain solutions to a generalization of the original problem:

> Let $p > 2$ be an integer. If $j \bmod p = 1$ or $j \bmod p = p - 1$, then $k = (j^2 - 1)/p$ is a base such that $(p1)_k$ is a perfect square j^2, provided that $k > p$.

This claim follows immediately from algebra. Calculations will be done in base 10.

Example: p = 3

Examine $j \bmod 3 = 1$ or $j \bmod 3 = 2$. Since $25 \bmod 3 = 1$, we can use $j = 25$ to obtain $k = 208$. Since $26 \bmod 3 = 2$, we can use $j = 26$ to obtain $k = 225$.

Example: p = 7

Examine $j \bmod 7 = 1$ or $j \bmod 7 = 6$. Since $15 \bmod 7 = 1$, we can use $j = 15$ to obtain $k = 32$. Since $20 \bmod 7 = 6$, we can use $j = 20$ to obtain $k = 57$. So $71_{32} = 15^2$ and $71_{57} = 20^2$.

Clarence W. Lienhard
clienhar@mansfield.edu
Mansfield University
Mansfield, PA, Jan. 7, 2014

IMPLICIT DIFFERENTIATION, THEN RECIPROCATION

Given the equation in x and y of a nonfunction relation, we can find dy/dx and dx/dy using implicit differentiation based on the chain rule. We more

$$\left(x+y^2\right)^3 = y$$

$$3\left(x+y^2\right)^2\left(1+2y\cdot\frac{dy}{dx}\right)=\frac{dy}{dx}$$

$$3\left(x+y^2\right)^2+6y\left(x+y^2\right)^2\frac{dy}{dx}=\frac{dy}{dx}$$

$$3\left(x+y^2\right)^2=\frac{dy}{dx}-6y\left(x+y^2\right)^2\cdot\frac{dy}{dx}$$

$$3\left(x+y^2\right)^2=\left(1-6y\left(x+y^2\right)^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{3\left(x+y^2\right)^2}{1-6y\left(x+y^2\right)^2}$$

Fig. 1 (Rogers)

$$\left(x+y^2\right)^3 = y$$

$$3\left(x+y^2\right)^2\left(\frac{dx}{dy}+2y\right)=1$$

$$3\left(x+y^2\right)^2\cdot\frac{dx}{dy}+6y\left(x+y^2\right)^2=1$$

$$3\left(x+y^2\right)^2\cdot\frac{dx}{dy}=1-6y\left(x+y^2\right)^2$$

$$\frac{dx}{dy}=\frac{1-6y\left(x+y^2\right)^2}{3\left(x+y^2\right)^2}$$

$$\frac{dy}{dx}=\frac{3\left(x+y^2\right)^2}{1-6y\left(x+y^2\right)^2}$$

Fig. 2 (Rogers)

commonly find dy/dx because we often require the slope of the line tangent to the graph of the relation at the point (x, y). To that end, we can first find dx/dy and then use the fact that

$$dy/dx = \frac{1}{dx/dy}.$$

This process benefits students in two ways. Not only does this process constitute an innovative and intuitive checking technique, but it also effectively doubles the drill value of each given problem.

Example

Find dy/dx for $(x+y^2)^3 = y$.

First, differentiate implicitly with respect to x (see **fig. 1 [Rogers]**).

Alternatively, differentiate implicitly with respect to y and then reciprocate (see **fig. 2 [Rogers]**).

We note that in the original equation, y was expressed more times than x, so finding dx/dy was "easier" than finding dy/dx.

An important benefit of this process for finding dy/dx by implicit differentiation and then reciprocation is solution efficiency. In general, always implicitly differentiate with respect to whichever variable, if either, is expressed more times in the relation equation than the other; then reciprocate as needed. Doing so can maximize efficiency, saving both time and effort and reducing the chance for careless error.

In addition, for problems requiring the slope of the line normal to the graph of the relation at the point (x, y), simply negate the sign of dx/dy.

James R. Rogers
rogers@bayou.com
Neville High School
Monroe, LA, Dec. 2, 2013

A CAVEAT ABOUT CURVE FITTING IN EXCEL

National Geographic (Jan. 2011) presented a graph of United Nations world population estimates and projections. Our experience in exploring these data in Microsoft Excel® brought home to us once again the importance of checking technological results, especially when powers of relatively large numbers are involved.

A scatter plot of the data (see **table 1 [Cibes and Greenwood]**, cols. 1 and 3) suggested higher-than-quadratic behavior (see **fig. 1 [Cibes and Greenwood]**). We looked at a cubic polynomial model in Excel using the given calendar years. We got a regression equation, "$y = -7E-06x^3 + 0.0412x^2 - 82.081x + 54452$," directly from the scatter plot (using the right mouse button and **Add Trendline** feature). As a check on this model, we used that equation to compute the populations (see **table 1 [Cibes and Greenwood]**, col. 4).

We were aware of potential problems with raising relatively large numbers such as calendar years to powers of 2 and above and did not expect perfect

Table 1 (Cibes and Greenwood)

World Population, Calendar Years 1930–2045

Year	Years since 1930	World Population Data (billions)	World Population Cubic Regression Models (billions)		
			EXCEL (using years)	EXCEL (using index years)	TI (using years)
1930	0	2	−821.8490	2.0028	2.0028
1960	30	3	−859.5920	2.9778	3.0015
1974	44	4	−877.1118	3.9145	3.9692
1987	57	5	−893.4668	4.9187	5.0159
1999	69	6	−908.7198	5.8787	6.0281
2011	81	7	−924.1971	6.7958	7.0109
2024	94	8	−941.3106	7.6567	7.9598
2045	115	9	−969.9029	8.5262	9.0119

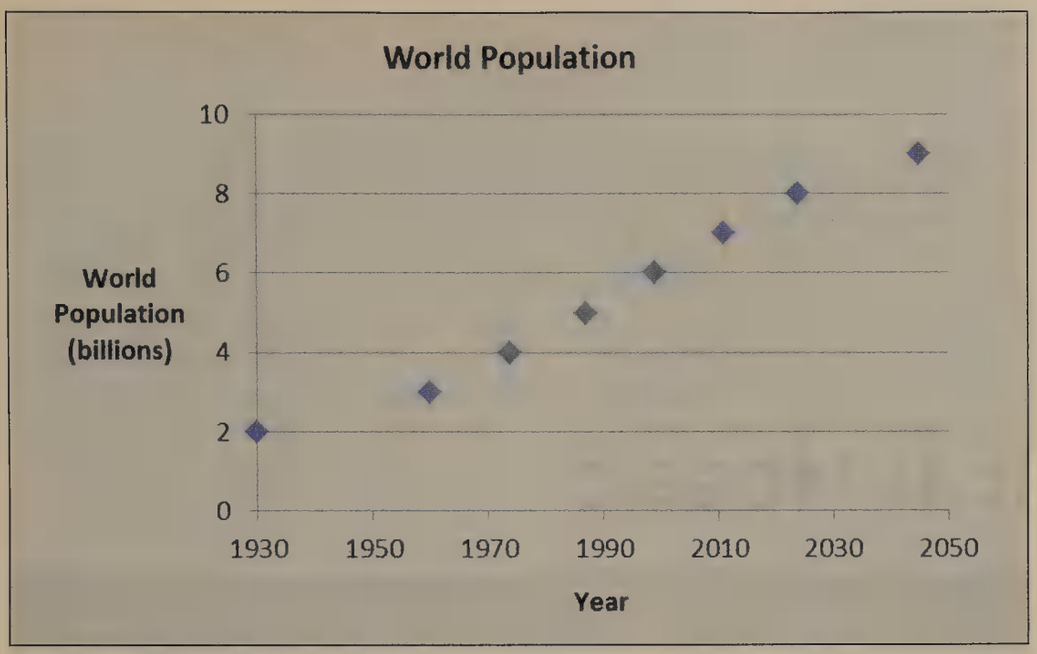


Fig. 1 (Cibes and Greenwood)

matches, but we did not anticipate that the computed values would be so far off, even negative. Looking back, we see a warning sign: The leading coefficient has only one significant digit and could have been as close to zero as -6.5×10^{-6}, which would have made the final values much less negative or even positive.

In an attempt to get more accurate Excel output, we replaced the calendar years with index years, as we have always taught students to do. Again, we got a regression equation, "$y = -7E - 06x^3 + 0.0013x^2 - 0.0002x + 2.0028$," directly from the scatter plot. This equation, with different coefficients for all but the leading term, gave population figures that were both positive and quite close to the given data (see **table 1 [Cibes and Greenwood]**, col. 5).

For another method, beyond using index years, of obtaining a more precise regression polynomial of degree 3 in Excel, see http://stackoverflow.com/questions/10857252/quadratic-and-cubic-regression-in-excel.

TI calculators display far more digits and allow for direct substitution into the regression equation through the VARS menu, so full use can be made of any extra digits that the calculator carries but does not always display. We entered the calendar year data into TI calculators and got coefficients that gave the regression equation $y = -6.887857 \times 10^{-6} x^3 + 0.0412048212 x^2 - 82.0811062x + 54451.89731$. Using VARS to substitute calendar years gave values that were closer to the given data than using Excel and index years (see **table 1 [Cibes and Greenwood]**, col. 6).

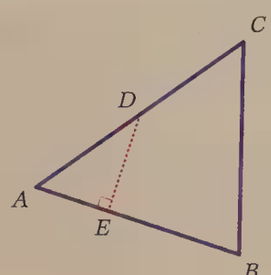
Margaret Cibes
cibesm@comcast.net
University of Hartford (retired)
James Greenwood
jhgreenwoodjr@gmail.com
University of Hartford (retired)
Hartford, CT, March 28, 2011

PROFESSOR SIEGEL APPENDS THE CALENDAR

Problem 8, October 2013

The Calendar problem for October 8, 2013 (*MT* Oct. 2013, vol. 107, no. 3, p. 200), was stated as follows:

> Acute triangle ABC has D as the midpoint of $\overline{AC}$. Let E be the foot of the perpendicular from D to $\overline{AB}$. If $AD = 5$, $AE = 4$, and $EB = 2AE$, find BC.



Here is an improvement to the offered solution (*MT* Oct. 2013, p. 203). There is no need to construct altitude CF. The cosine of angle BAC is $AE/AD = 4/5$. Then, as in *MT*'s solution, the law of cosines gives $(BC)^2 = 100 + 144 - 240(4/5) = 52$.

Department editor's note: "Great observation. This really simplifies the solution elegantly."

Problem 26, November 2012

The Calendar problem for November 26, 2012 (*MT* Oct. 2012, vol. 106, no. 3, p. 201), was stated as follows:

> In right triangle ABC, the bisector of $\angle ABC$ divides the opposite leg into two segments of length 5 and 4. Find the area of $\triangle ABC$.

MT's solution (*MT* Oct. 2012, p. 205) uses the angle bisector theorem.

Here is an alternate solution, which uses the double angle formula for the tangent function. We use the problem's figure. Designate the vertical altitude as h and the half angle at vertex B as b. Then $\tan b = 4/h$, and

$$\frac{9}{h} = \tan(2b) = \frac{2\tan b}{1-(\tan b)^2} = \frac{2(4/h)}{1-(4/h)^2}.$$

This gives $h = 12$. Thus, the area is $0.5(9)(12) = 54$.

Steven Siegel
siegel443@comcast.net
Niagara College (retired)
Lewiston, NY, various dates

MEDIA clips

Ron Lancaster

A Mathematical Mosaic

For the briefest time, Tim Van Horn put down roots at King and James. He parked his van, patted his two dogs, took out his camera and started snapping.

Van Horn, 44, is on a 10-year mission to make a people's portrait of Canada. He says 36,000 photos will populate his Canadian Mosaic. Finish date is 2017, Canada's 150th birthday. The photos will be assembled to make a 30-by-100-foot Canadian flag.

Source: "Smile, Hamilton, for a Special Portrait," by Kathy Renwald, *The Hamilton Spectator*, April 29, 2013, http://www.thespec.com/news/local/article/926040-video-hamilton-joins-a-portrait-of-canada

Since the beginning of the Canadian Mosaic project on October 1, 2008, 13,400 portraits from every province and territory in Canada have been created by photojournalist Tim Van Horn. Every year until 2017, Tim Van Horn will traverse the Canadian cultural landscape, interviewing and photographing thousands of Canadians, meticulously recording each individual story.

This project attempts to unite the country, a people, in one single all-encompassing, euphoric visual. Each one of us has a story to tell and to share. The Canadian Mosaic project brings our unique, interconnected stories together into one spellbinding visual. The Canadian Mosaic project's mission is to create an astonishing 36,000 portraits. This number represents a seemingly miniscule .1% of the population of Canada, but it will be the most comprehensive portrait ever created of the Canadian people. The collective story of these 36,000 portraits will converge and weave together a spectacular "prism of life," shining a spotlight on the meaning of life.

Source: Canadian Mosaic Project, http://canadianmosaic.ca/

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

Department editors

Louis Lim, *louis.lim1@gmail.com, Thornhill Secondary School, Thornhill, ON, Canada; and* ***Chris Bolognese,*** *cbolognese@uaschools.org, Upper Arlington City Schools, Columbus, OH*

1. **Table 1** shows 20 ways in which 36,000 photographs can be arranged in a rectangular shape. For example, Tim Van Horn could arrange the photographs in 80 rows with 450 photographs in each row. An arrangement that is not listed in the table is 2 rows with 18,000 photographs per row. How many arrangements are not listed in **table 1**?

Table 1 "A Mathematical Mosaic"

Number of Rows	Number of Photographs per Row
80	450
90	400
96	375
100	360
120	300
125	288
144	250
150	240
160	225
180	200
200	180
225	160
240	150
250	144
288	125
300	120
360	100
375	96
400	90
450	80

2. Suppose that Van Horn plans to take all 36,000 photographs in portrait mode and that the print size of each photograph will be 4 inches by 3 inches. Which of the arrangements in **table 1** could he use to make a mosaic 100 feet wide by 30 feet high?

3. Suppose that Van Horn takes the 36,000th photo on Canada's 150th birthday (July 1, 2017). Which of the following is closest to the number of days that it took Van Horn to complete the project: 2000, 3000, or 4000?

4. Use the estimate that you selected in question 3 to calculate the average number of photographs per day that Van Horn will need to take to finish the project on time.

5. When Van Horn began taking photographs, what did he estimate the population of Canada to be in 2017?

6. In 2008, Statistics Canada reported that the population of Canada in 2007 was 32,976,026 people and that it was expected to grow by about 0.9% per year until the year 2060. Use these data to calculate the population of Canada in 2017; compare your answer with Van Horn's estimate. (Source: Statistics Canada, http://www4.hrsdc.gc.ca/.3ndic.1t.4r@-eng.jsp?iid=35)

7. For each of the following scenarios, calculate the number of photographs Van Horn would need to take and the dimensions of the mosaic with the width-to-height ratio 100:30.

   (*a*) Van Horn's photographs represent 1% of the population of Canada in 2017.
   (*b*) Van Horn's photographs represent 10% of the population of Canada in 2017.
   (*c*) Van Horn's photographs represent 100% of the population of Canada in 2017.

8. The official flag of Canada (see **figs. 1** and **2**) consists of three vertical bands—red, white, and red—with an 11-point maple leaf in the center. The length-to-width ratio of the flag is 2:1. Van Horn's flag mosaic is not in the shape of the official flag of Canada, but suppose that he made a new version that was 100 ft. × 50 ft. How many 4 in. × 3 in. photographs would Van Horn need to fill this new design? How many more days would he need to take the photographs?

9. For a flag of area 200.00 cm^2, the area of the leaf (including the stem) measures 27.24 cm^2. Using these areas, estimate the percentage of photographs that would be inside the maple leaf for a mosaic in the shape of the official flag of Canada. How many photographs would fit inside the leaf in this mosaic?

Fig. 1 ("A Mathematical Mosaic")

SOURCE: HTTP://EN.WIKIPEDIA.ORG/WIKI/FILE:FLAG_OF_CANADA.SVG

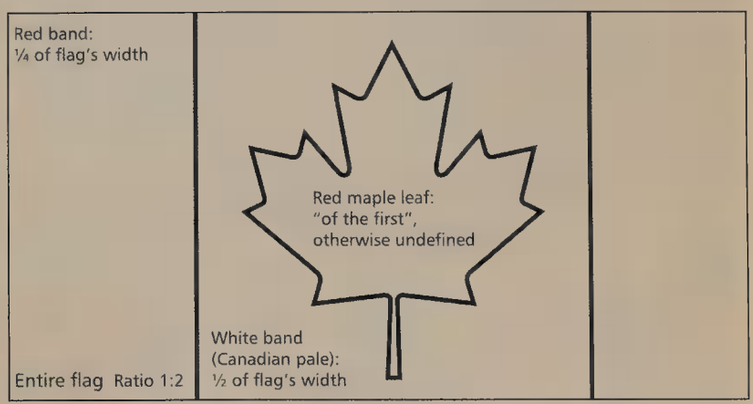


Fig. 2 ("A Mathematical Mosaic")

SOURCE: HTTP://EN.WIKIPEDIA.ORG/WIKI/FILE:CANADIAN_FLAG_CONSTRUCTION_SHEET.SVG

MEDIA CLIPS solutions

1. There are 72 factors of 36,000, so there are 72 ways in which the 36,000 photographs can be arranged in a rectangular shape. One way to see this is to write the prime factorization of 36,000. Since $36{,}000 = 2^5 \cdot 3^2 \cdot 5^3$, there are $(5 + 1)(2 + 1)(3 + 1)$, or 72, total positive integer factors. Therefore, 52 arrangements are not listed in **table 1**. **Table 2** contains a complete list of all possible arrangements of the photographs.

   The number 36,000 has an extremely large set of factors. The sum of the factors (not including 36,000) is equal to 91,764, which is greater than 36,000. Therefore, 36,000 is, by definition, an *abundant number*. An online calculator for determining the factors of a number and information about abundant numbers can be found at http://www.numberempire.com/36000 or http://en.wikipedia.org/wiki/Abundant_number.

2. Let n represent the number of photographs per row. Since the width of the flag is 100 ft. = 1200 in., we have $n \times 3$ in. = 1200 in.; thus, $n = 400$. The arrangement of the 36,000 photographs would have 400 photographs per row with a total of 90 rows.

3. Canada's 150th birthday will take place on July 1, 2017, and Van Horn started the project on October 1, 2008, so he will be working on the project for fewer than 9 years. There are $9 \times 365 = 3{,}285$ days (slightly more if we include leap days) in 9 years; thus, the closest estimate is 3,000 days.

   The exact number of days between October 1, 2008, and July 1, 2017 (including both dates) is 3,196. This number can be determined by manually counting the days by using the [redacted] (days between dates) function on the TI-84 calculator or at http://www.timeanddate.com/date/duration.html.

4. Van Horn will need to take an average of 12 photographs per day (36,000/3000).

Table 2 "A Mathematical Mosaic"

Number of Rows	Number of Photographs per Row	Number of Rows	Number of Photographs per Row
1	36,000	200	180
2	18,000	225	160
3	12,000	240	150
4	9,000	250	144
5	7,200	288	125
6	6,000	300	120
8	4,500	360	100
9	4,000	375	96
10	3,600	400	90
12	3,000	450	80
15	2,400	480	75
16	2,250	500	72
18	2,000	600	60
20	1,800	720	50
24	1,500	750	48
25	1,440	800	45
30	1,200	900	40
32	1,125	1,000	36
36	1,000	1,125	32
40	900	1,200	30
45	800	1,440	25
48	750	1,500	24
50	720	1,800	20
60	600	2,000	18
72	500	2,250	16
75	480	2,400	15
80	450	3,000	12
90	400	3,600	10
96	375	4,000	9
100	360	4,500	8
120	300	6,000	6
125	288	7,200	5
144	250	9,000	4
150	240	12,000	3
160	225	18,000	2
180	200	36,000	1

On his website, Van Horn had posted the number of photographs that he expects to take each year. **Table 3** shows a modified version of the data. Readers may graph these data and search for a pattern in the number of photographs taken per year.

5. Van Horn stated that the 36,000 photographs represent 0.1% of the population of Canada in 2017. If we let x represent the population of Canada in 2017, then

$$36{,}000 = 0.1\% \text{ of } x$$
$$36{,}000 = 0.001 \cdot x$$
$$x = 36{,}000/0.001$$
$$x = 36{,}000{,}000 \text{ people}$$

Table 3 "A Mathematical Mosaic"

Year	Number of Photographs Taken or to Be Taken
2008	256
2009	1,817
2010	3,164
2011	8,000
2012	5,500
2013	4,500
2014	3,500
2015	4,000
2016	3,500
2017	1,763
Total	**36,000**

6. According to the data provided by Statistics Canada, the population of Canada in 2017 will be $32{,}976{,}026 \cdot 1.009^{10}$, or approximately 36,000,000. It is not surprising that this number agrees with what Van Horn figured the population would be because any source that Van Horn used to make his estimate probably depended on data from Statistics Canada.

7. (*a*) To include 10 times as many photographs as the original project, we would need 10 times the amount of space. If we let k represent the linear scale factor, then

$$k^2 = 10$$
$$k = \sqrt{10}$$
$$k \approx 3.16227766.$$

Therefore, the dimensions of the mosaic for a scenario based on 1% of Canada's population will be about 316 ft. × 95 ft.

(*b*) The dimensions would be 1000 ft. × 300 ft., or 10 times bigger than the original mosaic.

(*c*) The dimensions would be 3162 ft. × 949 ft. (A mosaic of 36,000,000 photographs would be huge.)

8. The area of the new mosaic, 5000 ft.^{2}, would be 5/3 times the area of the original version proposed in question 2. The total number of photographs for the new design would be 5/3 of 36,000, or 60,000 photographs. Van Horn would need to take 14,000 additional photographs; doing so would take many more than 1000 additional days (or more than 3 additional years) if he took photographs at the same rate (12 photographs per day).

9. The area of the maple leaf is 13.62% of the area of the flag (27.24/200). Therefore, about 14% of the photographs—roughly 4,900 photographs of the 36,000 taken—would be inside the maple leaf. The exact number is difficult to determine because some of the photographs will not be completely inside the leaf.

RON LANCASTER, ron2718@nas.net, is a senior lecturer at the Ontario Institute for Studies in Education of the University of Toronto. His interests include photographs and videos of mathematics, math trails, magic, and puns.

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Seeing the Light, Part 2

Photograph 1
Door, YMCA Building, University of North Carolina at Chapel Hill

RON LANCASTER

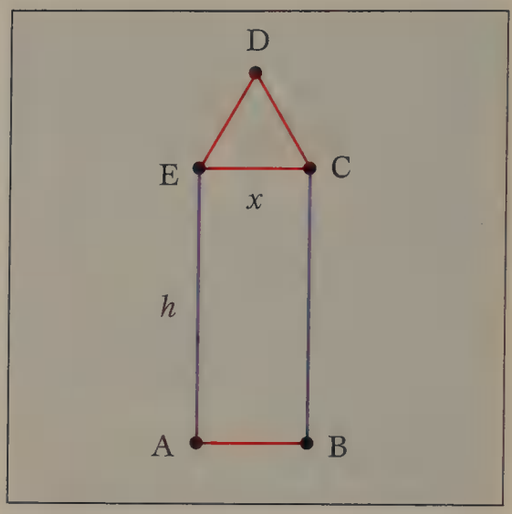


Fig. 1 A rectangular base is topped by an equilateral triangle.

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Department editors
Ron Lancaster, *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

The window shown in **photograph 1** is an unusual variation of a Norman window, the focus (pun intended) of the *MT* May 2014 Mathematical Lens (vol. 107, no. 9, pp. 652–55). This window inspired the editors of Mathematical Lens to examine a special case of Norman windows. We hope that we have opened the window for readers to search for more variations of Norman windows and that readers will submit photographs and questions for publication in *MT*.

1. A special case of the design that appears in **photograph 1** consists of an equilateral triangle *CDE* on top of rectangle *ABCE*. Suppose that the perimeter of the window is 8 meters. Let x represent the length (in m) of the side of equilateral triangle *CDE*, and let h represent the height (in m) of rectangle *ABCE* (see **fig. 1**).

   (*a*) Show that $h = (8 - 3x)/2$ for $0 \le x \le 8/3$.

   (*b*) Let y represent the combined area of triangle *CDE* and rectangle *ABCE*. Show that

   $$y = \frac{\sqrt{3}x^2}{4} + xh.$$

   (*c*) Use the equations from parts (*a*) and (*b*) to show that

   $$y = \frac{-\left(6-\sqrt{3}\right)x^2 + 16x}{4}.$$

   (*d*) Find the value of x for which y is a maximum. What are the

dimensions of the window whose area is the largest?

(*e*) For the window with the maximum area, show that the width-to-height ratio of the rectangle is approximately equal to 1.6 : 1.

(*f*) Which of the windows shown in **figure 2** most closely resembles the window with the maximum area?

2. In question 1, we considered y to be a function of x. But we can also think of y as a function of h.

(*a*) Express y as a function of h and find the value of h for which y is a maximum.

(*b*) Find the dimensions of the window whose area is a maximum. Compare these dimensions with those from question 1.

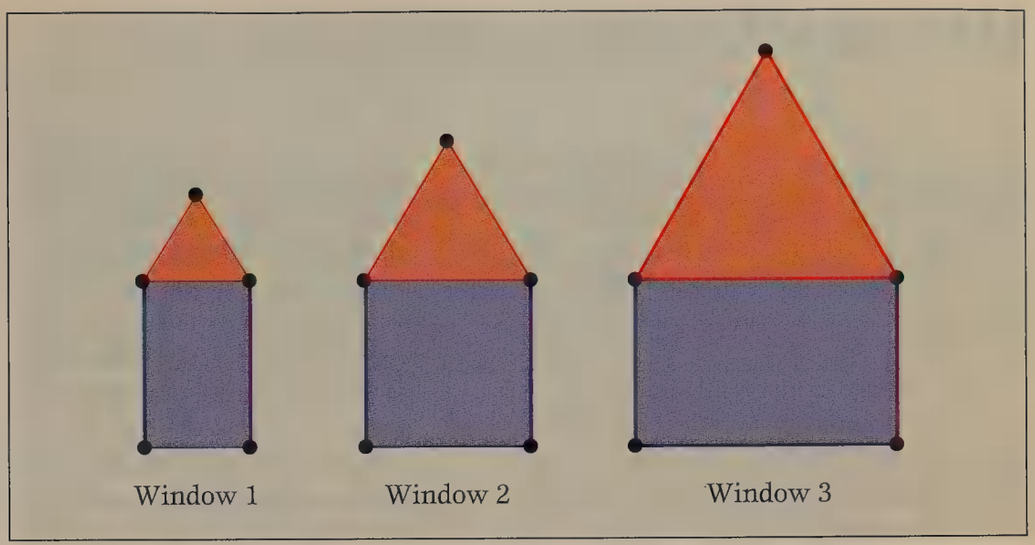


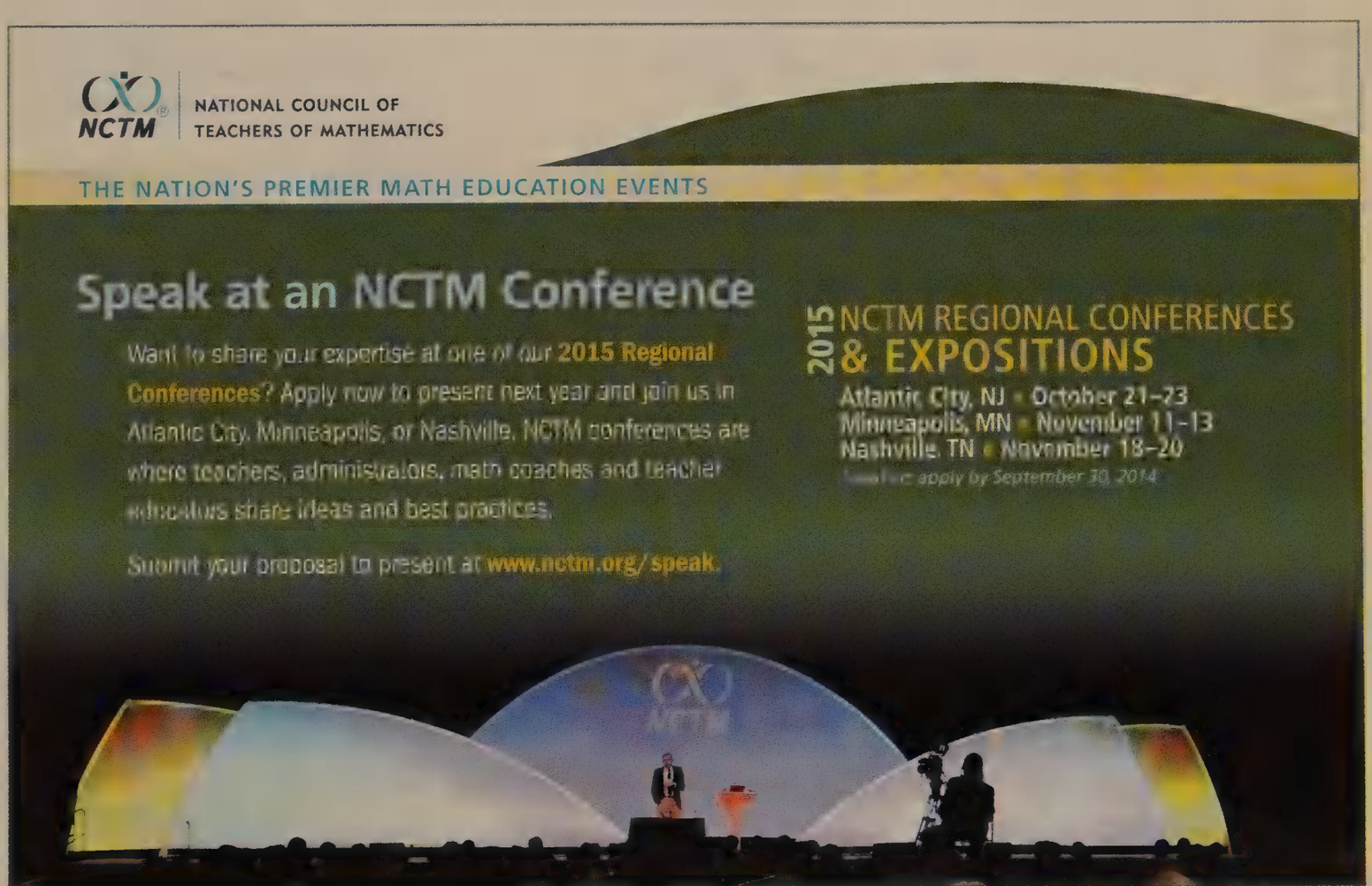
Fig. 2 Different base rectangles are topped by equilateral triangles.

3. Repeat questions 1 and 2 using the perimeter of the window as p m. The corresponding equation for question 1(*c*) is

$$y = \frac{-\left(6-\sqrt{3}\right)x^2 + 2px}{4}.$$

The corresponding ratio for question 1(*e*) is

$$\frac{2\left(6+\sqrt{3}\right)}{3\left(5-\sqrt{3}\right)}.$$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MATHEMATICAL LENS solutions

1. (*a*) Note that $3x + 2h = 8$ and, therefore, $h = (8 - 3x)/2$. Since $h \geq 0$, $8 - 3x \geq 0$ and so $x \leq 8/3$. The length of a line segment is always positive; therefore, $0 \leq x \leq 8/3$.

(*b*) The area of an equilateral triangle with side length s is equal to $\sqrt{3}s^2/4$, and the area of a rectangle is the base times the height. Therefore,

$$y = \frac{\sqrt{3}x^2}{4} + xh.$$

(*c*) Using the equations from parts 1(*a*) and 1(*b*), we have the following:

$$\begin{aligned} y &= \frac{\sqrt{3}x^2}{4} + xh \\ &= \frac{\sqrt{3}x^2}{4} + x\left(\frac{8-3x}{2}\right) \\ &= \frac{\sqrt{3}x^2}{4} + \frac{8x-3x^2}{2} \\ &= \frac{\sqrt{3}x^2 + 16x - 6x^2}{4} \\ &= \frac{-\left(6-\sqrt{3}\right)x^2 + 16x}{4} \end{aligned}$$

(*d*) Note that y is a quadratic function of x. The vertex of the parabola can be found by means that apply specifically to quadratic equations, such as completing the square or finding the point midway between the x-intercepts of the parabola. Alternatively, the graph of y versus x (see **fig. 3**) can be used to find that the value of x is approximately 1.87. Students who are studying calculus might use the derivative to find the vertex.

The x-value 1.87 is between 0 and 8/3 and thus lies within the domain of x. Since $h = (8 - 3x)/2$, it follows that h is approximately 1.18. Therefore, the area is a maximum when the length of the side of equilateral triangle CDE is 1.87 m and the height of rectangle $ABCE$ is 1.18 m.

(*e*) The ratio 1.87/1.18 is approximately equal to 1.6 as required.

(*f*) Window 3 most closely resembles the window with the maximum area.

2. (*a*) Note that $3x + 2h = 8$ and, therefore, $x = (8 - 2h)/3$. Since $x \geq 0$, $8 - 2h \geq 0$ and so $h \leq 4$. The length of a line segment is always positive; thus, $0 \leq h \leq 4$. To find x and y in terms of h, we write $3x + 2h = 8 \rightarrow x = (8 - 2h)/3$ and

$$y = \frac{\sqrt{3}}{4}\left(\frac{8-2h}{3}\right)^2 + \left(\frac{8-2h}{3}\right)h.$$

The graph of y versus h is shown in **figure 4**. The value of y is a maximum when h is approximately equal to 1.18. The value 1.18 is between 0 and 4 and thus lies within the domain of h.

(*b*) Since $x = (8 - 2h)/3$, it follows that x is approximately 1.87. Therefore, the area is a maximum when the height of rectangle $ABCE$ is 1.18 m. and the length of the side of equilateral triangle CDE is 1.87. The dimensions of the window are the same as in question 1. Problems can be solved in many ways, but if a problem has only one solution, then all valid methods must produce the same answer.

3. Using the perimeter p, we obtain $h = (p - 3x)/2$, where $0 \leq x \leq p/3$, and

$$y = \frac{-\left(6-\sqrt{3}\right)x^2 + 2px}{4}.$$

Writing $y = ax^2 + bx + c$, we find that the maximum area y occurs at

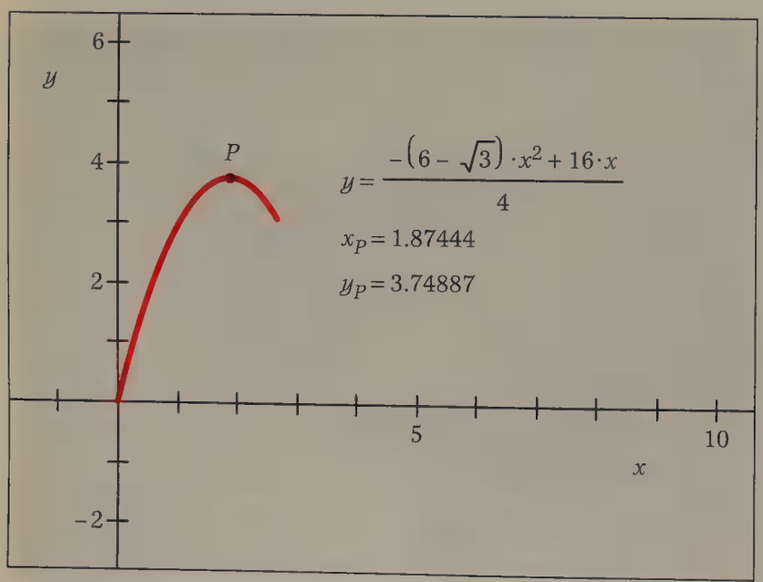


Fig. 3 The maximum area, y, occurs where the window width, x, is about 1.87.

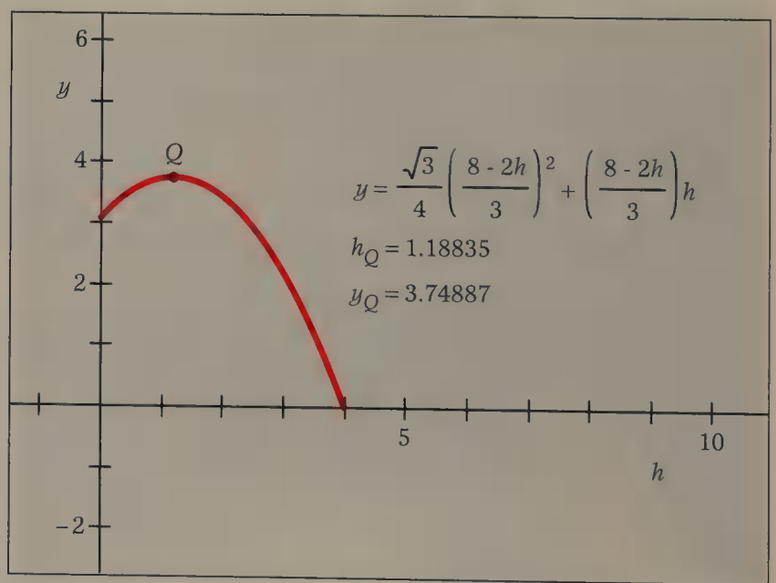


Fig. 4 The maximum area, y, occurs where the rectangle's height, h, is about 1.19.

$$x = \frac{-b}{2a} = \frac{-2p/4}{2\left(-\left(6-\sqrt{3}\right)/4\right)} = \frac{p}{\left(6-\sqrt{3}\right)}.$$

It follows that the height of rectangle $ABCE$ is $h = p(5 - \sqrt{3})/22$ m. The width-to-height ratio of the rectangle is equal to

$$\frac{x}{h} = \frac{p\left(6+\sqrt{3}\right)/33}{p\left(5-\sqrt{3}\right)/22} = \frac{2\left(6+\sqrt{3}\right)}{3\left(5-\sqrt{3}\right)} \approx 1.6.$$

$$\text{solve}\left\{\frac{\sqrt{3}}{2}\cdot\frac{p-2\cdot h}{3}\cdot\frac{-2}{3}+\frac{p-2\cdot h}{3}+\frac{-2}{3}\cdot h=0,h\right\} \qquad h=\frac{-p\cdot\left(\sqrt{3}-5\right)}{22}$$

Fig. 5 CAS solves for h in terms of p.

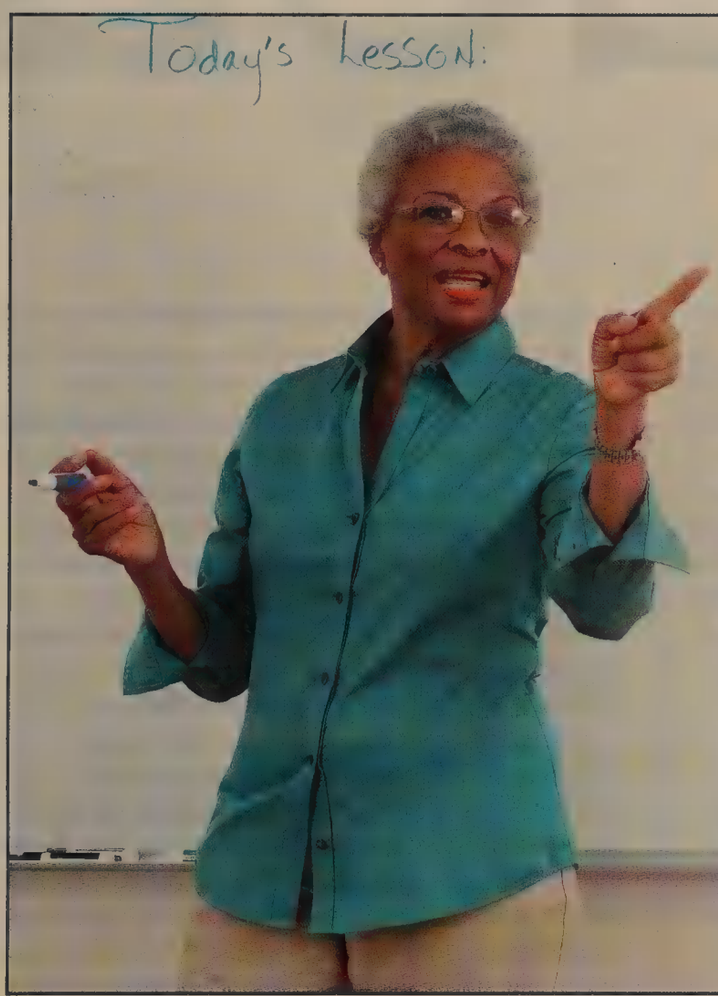
The second problem could be addressed using derivatives and CAS. With $x = (p - 2h)/3$ and $0 \le h \le p/2$, we have

$$y = \frac{\sqrt{3}}{4}\left(\frac{p-2h}{3}\right)^2 + \left(\frac{p-2h}{3}\right)h.$$

Thus,

$$\frac{dy}{dh} = \frac{\sqrt{3}}{2}\left(\frac{p-2h}{3}\right)\left(\frac{-2}{3}\right)+\left(\frac{p-2h}{3}\right)+\left(\frac{-2}{3}\right)h.$$

Using the TI-Nspire (see **fig. 5**), we can see that $h = -p(\sqrt{3} - 5)/22$. This is the same answer that was obtained previously. Note that $p(5 - \sqrt{3})/22$ is greater than 0 and less than $p/2$ and lies within the domain of h.



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CALL FOR MANUSCRIPTS

Alfinio Flores

Who's in the Lea

The striking results of this coin-tossing simulation help students understand the law of large numbers.

Tossing a fair coin 1000 times can have an unexpected result. In the activities presented here, players keep track of the accumulated total for heads and tails after each toss, noting which player is in the lead or whether the players are tied. The winner is the player who was in the lead for the higher number of turns over the course of the game. Students are in for many surprises: The numbers of ties along the way is far lower than they expected; it is not unlikely that one player will be in the lead a very high percentage of the time (Peterson 1998, p. 16); and students see that two different measures—the relative frequency of heads (or tails) and the portion of times in the lead—can behave very differently. Students learn that the law of large numbers does not imply any regularity in the outcomes in the short term (such as frequent alternations of tails and heads or frequent ties).

The law of large numbers states that as the number of trials increases, the experimental probability is more likely to be close to the theoretical probability. Thus, in the long run, when flipping coins, we may reasonably expect that the relative portion of heads (or tails) will be close to the theoretical probability. However, one common misconception is believing that for short sequences of random trials the empirical probability will also be close to the theoretical probability. This misconception is sometimes referred to as "the belief in the law of small numbers" (Tversky and Kahneman 1971).

The activities presented here help students become aware of such misconceptions, make them explicit, and confront and correct them. These activities are not meant to be a first introduction to the law of large numbers. Rather, their purpose is to help students develop a deeper understanding of what the law does and does not say. We add a layer of complexity by keeping track of outcomes in two

ways—recording not only the outcome of the toss but also who is in the lead after each toss.

THE GAME

One player is assigned heads, the other tails. After each toss of the coin, students write down the outcome and the totals accumulated for heads and tails. The player who has the higher accumulation—that is, the one who is in the lead—receives a point. If players are tied, no one receives a point. Thus, being in the lead at a given time is determined not only by the last toss but also by previous tosses. The winner of this game is the player who was more often in the lead, not necessarily the player who had the higher portion of successful flips at the end of 1000 tosses. **Table 1** shows the results for the beginning of a game in which the first five tosses are HTHTT. Here, H was in the lead two times, T was in the lead once, and the players were tied twice.

BRENDA MCEWAN/GETTY IMAGES

Guessing before the Experiment

Before the activity is conducted, students are asked to predict, for 1000 tosses, what percentage of the time each player will be in the lead and how many times the players will be tied; then students enter their estimates in the first row of **table 2** (they will fill in the second row later). When students articulate their ideas before the experiment or analysis, it is more likely that an unexpected result will cause them to rethink their basic assumptions. "To correct misconceptions, it is useful for students to make predictions and then compare the predictions with actual outcomes" (NCTM 2000, p. 254).

Table 1 Results from 5 Tosses

	Toss Outcome	Accumulation H	Accumulation T	Who's in the Lead?
1	H	1	0	H
2	T	1	1	Tie
3	H	2	1	H
4	T	2	2	Tie
5	T	2	3	T

Table 2 Estimations before the Experiment of 1000 Tosses

	Number of Times the Winner Will Be in the Lead	Number of Times the Loser Will Be in the Lead	Number of Times the Players Will Be Tied
First guess			
Second guess			

PROGRAM: HT40

```
AxesOff                 No axes
FnOff                   No functions graphed
ClrDraw                 Screen is cleared
For(N,1,4)              Start loop of four rows
For(M,1,10)             Start loop for each row
If rand <.5             A random number is generated and compared
                          with .5
Then                    When random number is less than .5
Text(6*N-6,6*M,"T")     Outcome of T displayed on the screen
Else                    When random number is not less than .5
Text(6*N-6,6*M,"H")     Outcome of H displayed on screen
End                     End of if-then-else block
End                     End of each row loop
End                     End of four-row loop
```

Fig. 1 The program output is formatted to show four rows of ten characters, H or T.

Conducting the Experiment

Students complete two experiments using calculator simulation programs, first for a game of 40 coin tosses (see **fig. 1**) and then of 1000 (see **fig. 2**). If the students were to carry out the first experiment by hand, tossing a coin repeatedly, 40 is a number small enough for them to record the results and look for patterns in just a few minutes. At the same time, 40 tosses will in many cases show results strikingly contrary to students' expectations of how often one of the players will be in the lead.

For a sample size of 1000, it is very likely that the relative frequency of heads or tails will be very close to .5, illustrating the law of large numbers.

These short programs written for graphing calculators show the commands in the first column and descriptions of what the commands do in the second column. If students do not have access to graphing calculators, tools such as Mathematica®, Excel®, Fathom Dynamic Data™ Software, TinkerPlots®, and the National Library of Virtual Manipulatives would be appropriate for simulating random phenomena.

SIMULATING THE GAME

40 Tosses

For this activity, students will work in pairs to simulate 40 tosses of a coin. One student chooses heads and the other tails. Before students run the simulation program, have them predict what proportion of the time each of the players will be in the lead and how many times the players will be tied. Have students justify their guesses—that is, make explicit the reasons for the final numbers they predicted. Then students run the program HT40 (see **fig. 1**) on the TI-84 graphing calculator. The program generates a random sequence of H and T, with the same probability, and lists them sequentially on the calculator screen. Players can use an expanded version of **table 3**, with 40 rows instead of 5, and fill in the first column of the table with the results. In the next two columns, they write the accumulated number of tosses for H and for T. In the last column, they record who was in the lead after each toss or whether the players were tied. Have students count how many times each player was in the lead and how many times the players were tied.

Students can answer questions such as these:

- How many times was H in the lead? What is the ratio of the number of times that H was in the lead to the total number of tosses?
- How many times was T in the lead? What is the ratio of the number of times that T was in the lead to the total number of tosses?
- How many times were the players tied?

- Compute the relative frequency for heads and the relative frequency for tails. How close are these ratios to .5?

Students can compare the results with their predictions of how often one player would be in the lead. They can discuss with their partner and explain the results in their own words. Students can share with the whole group their results and what they notice. For example, one pair noticed that in their simulation, even though the ratio of accumulated number of tails to the total was 21/40 (52.5%), T was in the lead 32 out of 40 times (80%).

1000 Tosses

The program LEADING (see **fig. 2**) simulates tossing a coin 1000 times and keeps track of whether H or T is in the lead. This program does not display the random sequence of outcomes H and T. Instead, it displays who is in the lead at a given time and how many times that player has been in the lead so far. At the end, it will display the total accumulated number of H and T. At that point, the program will pause. To display the final results of how many times each one was in the lead, press ENTER.

Before students run the program, have them estimate again, on the basis of their experience with the previous activity, how many times each player will be in the lead and how many times the two will be tied if the game consists of 1000 tosses (see **table 2**, second row). Then have students run the program, observe the display, and describe what they notice. Even though the results are displayed quite rapidly, students should be able to notice something striking in the displays of who is in the lead.

PROGRAM: LEADING

```
0→H                    Count of H is set to zero
0→I                    Count of times H is in the lead is set to zero
0→T                    Count of T is set to zero
0→S                    Count of times T is in the lead is set to zero
For (N,1,1000)         Loop of 1000 tosses starts
If rand <.5            A random number is generated and
                         compared to .5
Then                   When number is less than .5
H+1→H                  The number of H is increased by 1
Else                   When number is not less than .5
T+1→T                  The number of T is increased by 1
End                    End of if-then-else block
If H>T                 Start of block to determine who is in the lead
Then                   If H is in the lead
I+1→I                  Number of times H is in the lead is
                         incremented
Disp "H",I             Display H and the number of times that
                         H has been in the lead
Else                   When H is not in the lead
If T>H                 (begin nested block) Check if T is in the lead
Then                   When T is in the lead
S+1→S                  Number of times that T is in the lead is
                         incremented by 1
Disp "T",S             Display T and the number of times that
                         T has been in the lead
End                    End nested if-then block
End                    End if-then block
End                    End loop thousand tosses
Disp "H", H            Display total accumulated number of H
Disp "T", T            Display total accumulated number of T
PAUSE                  Pause program
Disp "H LEADING", I    Display number of times H was in the lead
Disp "T LEADING", S    Display number of times T was in the lead
Disp "TIES", 1000 - I - S  Display number of ties
```

Fig. 2 Four counters are used to accumulate the numbers of heads (H), tails (T), times that heads are in the lead (I), and times that tails are in the lead (S).

STUDENTS' RESULTS

Students' Predictions

Misconceptions about probabilistic experiments are fairly common, even among people who have strong mathematical backgrounds. To illustrate, I will share the predictions of thirty-nine college students who were asked to estimate how often each player would be in the lead and how often the two players would be tied for 1000 tosses of a coin. These students had strong mathematics backgrounds, including a college course in probability.

A vast majority (77%) of the students expected a high number of ties (more than 150). In fact, thirteen of the thirty-nine students (33%) expected the players to be tied 500 times—the maximum number possible of ties. Only eight students (21%) expected 100 or fewer ties. Few students predicted that the winner would be in the lead for a large portion of the 1000 trials. Only seven students (18%) predicted that the winner would be in the lead 700 or more times, whereas twenty-six students (67%) predicted that the winner would be in the lead 400 or fewer times. Most students expected the loser not to be too far behind the winner. Only one student predicted that the loser would be in the lead fewer than 100 times.

Table 3 Outcomes of Coin Tosses (first five rows only)

	Toss Outcome	Accumulation H	Accumulation T	Who's in the Lead?
1				
2				
3				
4				
5				

Table 4 Outcomes for 1000 Tosses

Event	Student Predictions	Results from 100 Simulations
The number of ties was fewer than 100.	21%	100%
The winner was in the lead 700 times or more.	18%	75%
The winner was in the lead 400 times or fewer.	67%	0%
The loser was in the lead fewer than 100 times.	3%	47%

As we see in **table 4**, the predictions made by students differ starkly from what actually happens when random trials of tossing fair coins are simulated using technology. Students' predictions, especially those predicting a high number of ties, are a strong indication of their belief in the law of small numbers. For example, the only way to have 500 ties in 1000 tosses (as predicted by one-third of the students) is if the ratio of heads (or tails) to the total in the empirical results stays very close to the theoretical probability, for every segment of the experiment. High school students and their future teachers must have the opportunities to correct such misconceptions about probability.

Simulation Results

The results of 100 random simulations of games of 1000 tosses are given in **table 4**. In all cases, the number of ties was fewer than 100. The winner was in the lead 700 times or more in 75% of the simulations, and in 47% the loser was in the lead fewer than 100 times. Contrast students' predictions with the calculator simulations. We can see that they are remarkably different. As the students were running the program, they were surprised by the long periods of time when one player was in the lead. In one run of the program, although the final accumulation was 500 H and 500 T, the lead was tied only 36 times, and the lead was held 791 times by T but only 173 times by H. In another simulation, T was in the lead the entire time, even though the final ratios of H and T to the total did not differ that much (0.487 for H, 0.513 for T). In a third run, T was in the lead 946 times and H only 33 times, but the accumulated total number of H was higher than that of T (503 H and 497 T).

CONFRONTING MISCONCEPTIONS

Students were startled by the results of the calculator simulations. They realized that they were expecting too much regularity in the short term—that is, they were expecting frequent alternations of heads and tails without any long runs of heads or tails. However, once faced with the results, students were able to explain the long periods of time when one player was in the lead. One student commented that once a player has a considerable advantage, because the theoretical probability for the next tosses is 50-50, that player will most likely preserve the lead for a long time. In other words, a well-established lead is difficult to overcome.

In contrast, the relative frequency of heads (or tails) will likely approach the theoretical probability as the number of tosses increases, even when one player has an initial advantage. For example, if the first 10 tosses resulted in 7 heads and the next 100 tosses resulted in 52 heads, the combined results give a ratio of 59/110 = 0.536..., which is closer to 1/2 than 7/10, even though the new tosses resulted in still more heads than tails.

Experiences like the one described here help students understand better the behavior of random phenomena. After this coin-tossing activity, students understood better why a player with a considerable advantage will likely retain the advantage for a long time. The accumulated results are seldom tied, and the lead does not change frequently. Different measures of data from randomly generated phenomena can behave very differently from one another. Nonroutine, technology-rich tasks such as those described here have the potential to challenge and upend students' misconceptions about probability as well as provide rich opportunities for them to develop robust probabilistic reasoning skills.

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ALFINIO FLORES, alfinio@udel.edu, teaches mathematics and mathematics methods courses *con ganas* at the University of Delaware in Newark. He uses technology to help teachers and students develop a better understanding of mathematical concepts.

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6 Principles for Quantitative Reasoning and Modeling

Eric Weber, Amy Ellis, Torrey Kulow, and Zekiye Ozgur

Modeling the motion of a speeding car or the growth of a Jactus plant, teachers can use six practical tips to help students develop quantitative reasoning.

Encouraging students to reason with quantitative relationships can help them develop, understand, and explore mathematical models of real-world phenomena. But what is quantitative reasoning?

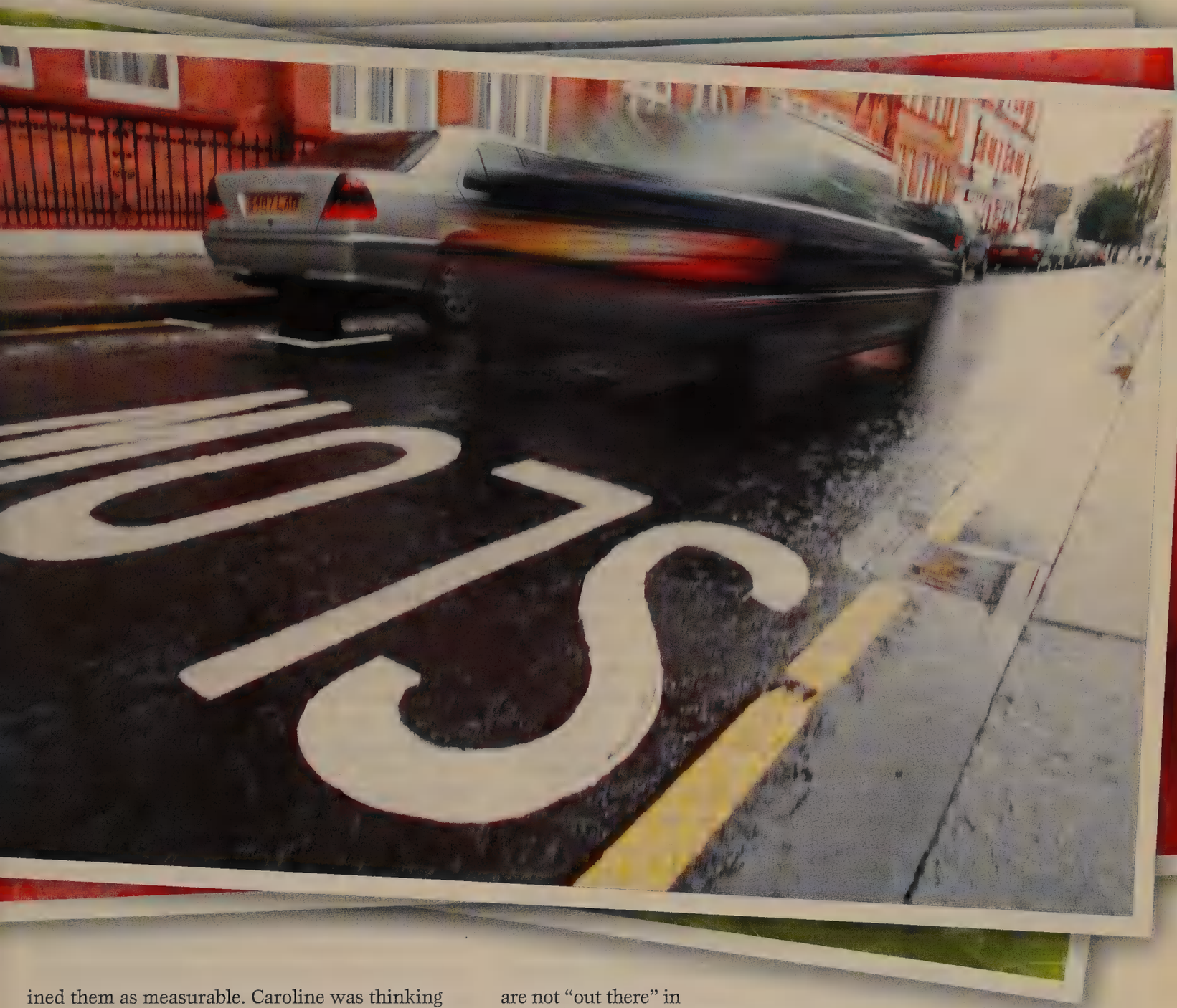
Consider an example. Study the picture shown in the **opening photograph**. What do you see when you look at the picture?

Imagine that a teacher gave that picture to his or her students and asked them to describe and write down everything they saw in the picture. The following three students examine the figure and focus on different aspects:

- Amelia notes that there are two cars, a marker that says "slow," and some buildings in the background.
- Ben notices that the blue car is moving and that the silver car is parked. He also notes that there is a "space" between the two cars, but he does not think any further about how big the space is or how it could be measured.
- Caroline focuses on the length of the blue car compared with that of the silver car, wondering which is greater in terms of feet. She notices that the blur in the picture is caused by the car moving some distance in the amount of time that the camera's shutter was open.

Amelia's conception of the scene is what we would call *non-quantitative*; she did not focus on any aspects of the figure that could be quantified. Ben's conception could be called *proto-quantitative*; how he thought about the space between the two cars is necessary for building a quantitative conception of the situation without actually conceiving of any quantities.

Caroline, in contrast, noticed attributes of the situation, such as the length of each car, and imag-



ined them as measurable. Caroline was thinking quantitatively; for instance, one thing she focused on was how far the car might have moved during the time when the camera's shutter was open. If Caroline were to develop a general way to express average speed, she would need to conceive of a change in one quantity (the distance that the car traveled) in relation to another quantity (the number of seconds that the shutter was open).

QUANTITIES AND QUANTITATIVE REASONING

A *quantity* is a measurable attribute of an object or phenomenon. *Quantitative reasoning* is a way to describe the mental actions of a student who conceives of a mathematical situation, constructs quantities in that situation, and then relates, manipulates, and uses those quantities to make a problem situation coherent. Thompson (2011) argued that quantities are not "out there" in a problem situation; instead, quantities exist in students' conceptions of situations, and their measurement requires quantification.

Quantification (Thompson 2011) is the process of conceptualizing an object (such as a car) and an attribute of the object (such as how far the car traveled) so that the attribute has a unit of measure (such as feet). This process of quantification is essential to constructing a quantity.

Consider a student's understanding of speed. To imagine assigning a speed to an object, he or she must first have conceived of speed as a quantification of completed motion, which entails that there is something about the object's motion that can be measured (Thompson and Thompson 1994). Thompson and Thompson argued that to quantify completed motion, the student must imagine a way to measure it. The measurement of speed relies on

conceptualizing two quantities—the distance traveled and the time required to travel that distance—and a multiplicative comparison of those quantities. Then the student must be able to conceive of a value resulting from the measurement process. In this case, speed represents a direct proportional relationship between the distance traveled and the time required to travel that distance. The *value* is the result of the quantification process, such as 55.2 miles per hour. Understanding speed in this way results from using models of a situation (such as a car traveling) and representing its measurable components.

We believe that quantitative reasoning is the foundation that supports students in modeling phenomena mathematically because it provides a means for them to create relationships between quantities that constitute those models. We see quantitative reasoning as a category different from forms of inquiry such as discovery-based learning. It is not a teaching technique per se but a specific way of thinking about mathematics. In other words, a focus on quantities and their relationships means approaching the mathematical topics themselves from a specific mathematical stance, one that emphasizes activities such as measurement, conceptualizing magnitudes, and constructing mathematical relationships.

Because we see quantitative reasoning as a mathematical orientation, we think it is broadly applicable to all teachers interested in supporting modeling in their classrooms, regardless of the type of inquiry they use. Here we describe the relationship between modeling and quantitative reasoning, provide principles for integrating quantitative reasoning in the classroom, and offer examples to demonstrate the utility of these principles for supporting students' mathematical modeling activities.

MODELS AND MODELING

Models are conceptual structures that represent real or imagined scenarios and relate elements of those situations. *Modeling* involves building, testing, and applying conceptual representations of phenomena—a practice that is central to learning and doing science and mathematics. Modeling activities that are grounded in students' conceptualization of quantities are a powerful way to support students' reasoning and sense making as they explore mathematical ideas and relationships.

Both the NextGen Science Standards (2013) and the Common Core State Standards for Mathematics (2010) point to modeling as a central, unifying theme across science and mathematics. Instruction based on modeling begins with the goal to engage students in understanding the world by constructing and using models to describe, explain, and predict

The Six Principles

1. Rewrite a problem situation or prompt so that the students must identify the quantities that they believe are relevant to solving the problem.
2. Ask questions about a problem that focus on why students chose to identify particular quantities and how they intend to or imagine measuring those quantities.
3. Have students identify and test relationships between the quantities that they measure. Push them to justify why those relationships always or do not always hold.
4. Once students have created a model, ask them to determine how varying individual quantities affects the rest of the quantities in the model.
5. Have students develop a representation (physical, visual, etc.) of the situation they are modeling that consists of all the quantities and their interrelationships.
6. Have students revise and retest aspects of their model that may not have been accurate.

Fig. 1 Teachers can use these practical tips to integrate quantitative reasoning into modeling lessons.

the behavior of physical or hypothetical phenomena. We have found the principles shown in **figure 1** to be useful in thinking about how to integrate quantitative reasoning in modeling instruction.

IMPLEMENTING THE PRINCIPLES

In this section, we elaborate on the six practical suggestions shown in **figure 1** by describing a scenario from a teaching and research setting. This scenario and others like it demonstrate how a relatively typical problem (the growth of a Jactus plant), and the medium used to present it (a GeoGebra script) can be articulated from a quantitative reasoning perspective and used to support students' modeling activities.

The Problem Context Affords Quantification (Principles 1, 2, 5)

The Jactus plant context (see **fig. 2**) invites students to explore the growth of a plant that increases its height by a constant multiplicative factor (such as doubling) each week. A specially designed GeoGebra script (see **fig. 3**) enables students to explore how the plant grows exponentially over time by changing the parameters (such as the growth factor and the initial height) and then dragging the plant's base (go to **www.nctm.org/mt056**). Although the Jactus scenario is not a realistic problem, it provides an easily visualized context with quantities that students can directly observe and manipulate, in contrast to scenarios such as population growth or compound interest. As students explore the way in which the plant grows, teachers can ask them to

The Jactus Plant Problem

Here is a picture of the Jactus plant. Compare to this picture and draw another picture of what the Jactus will look like after 1 week. Then draw a picture of what the Jactus will look like after 3 weeks.

Fig. 2 A growth problem uses quantities that students can observe and manipulate.

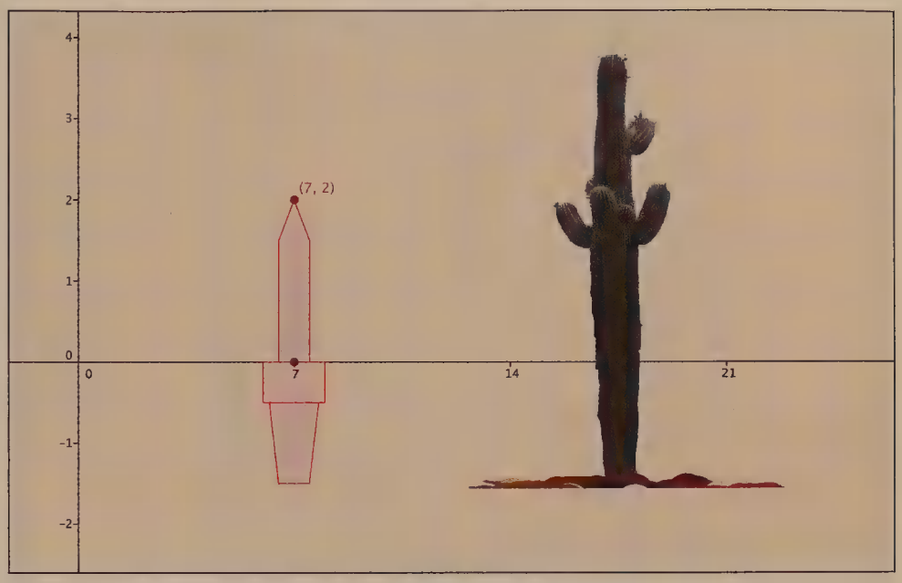


Fig. 3 A GeoGebra script allows for visualization of quantities.

think about the following questions: "What variables do you think might contribute to the Jactus's growth?" and "How can you keep track of how fast the plant is growing?"

These open-ended prompts encourage students to identify the relevant quantities in determining how the plant grows (see principle 1). Students often mention multiple factors that influence a plant's growth, such as sunlight, fertilizer, and soil quality, but will begin to isolate the relevant quantities height (in inches) and time (in weeks) when tasked with tracking the plant's growth. Asking students to think about how they can keep track of growth encourages them to think about how that growth can be measured over time (see principle 2). Asking students to create a picture or diagram of the plant (see principle 5) can also provide a way to assess how they are thinking about the relationship between height and time.

Student Models Help Refine Concepts (Principles 4 and 6)

The modeling task (see **fig. 2**) was introduced to students during a five-week summer school course for incoming ninth-grade students who were preparing to enroll in algebra in the fall. In this course, students met for seventy-five minutes a day and explored three function families—linear, exponential, and quadratic. In small groups of two to four, students worked together on new problems and frequently shared their ideas with one another and with their teacher in a whole-class discussion. All the students in the course had completed general eighth-grade mathematics or prealgebra, but none had experienced exponential functions before the summer school course.

On the second day of the exponential functions unit, before they had encountered equations of the form $y = 2^x$, the students explored a GeoGebra script for a plant with an initial height of 1 inch that doubled each week. They were asked to draw a picture of the plant after one week and after three weeks. **Figure 4** shows Juan's drawing (**a**) and Paj's drawing (**b**). The drawings reveal that Paj tried to capture the plant's height at three weeks as quadruple its height at one week, whereas Juan was not understanding the growth as exponential.

To model the Jactus plant's growth, we must isolate the two quantities, height and time; determine that height increases multiplicatively as time increases additively; and ultimately coordinate the height ratio of y_2 to y_1 for corresponding x-values representing time for any Δx. For instance, Paj appeared to be able to visualize the height of the Jactus at week 3 as four times as large as the height at week 1, a two-week difference for the function $y = 2^x$. Questions that encourage this coordination are ones that ask students to construct and model relationships between height and time for different Jactus plants (see principle 3). For instance, Paj constructed a table (see **fig. 5**) to represent the growth of the Jactus plant that she drew in **figure 4b**, and Evan represented the relationships in Paj's table algebraically by writing "height = 2^{week}".

Once students identify repeated multiplication as the mechanism for determining the plant's height, they can then begin to think about how varying the initial height and the growth factor will affect the Jactus's growth (see principle 4). Teachers could ask students to explore a new plant with an initial height of 1/2 inch, rather than 1 inch, and compare the growth of the new and old plants. By creating tables to model the growth of both plants, students can see that the old plant is twice as tall as the new plant at every week value.

This process led one student, Bonito, to revise the previous equation that the students had developed, $y = b^x$, into the form $y = a \cdot b^x$ to account for the initial height (see **fig. 6**). After this discussion, the students always included the initial height, even if it were 1 (principle 6). Encouraging students to justify their representations and to explain what quantity

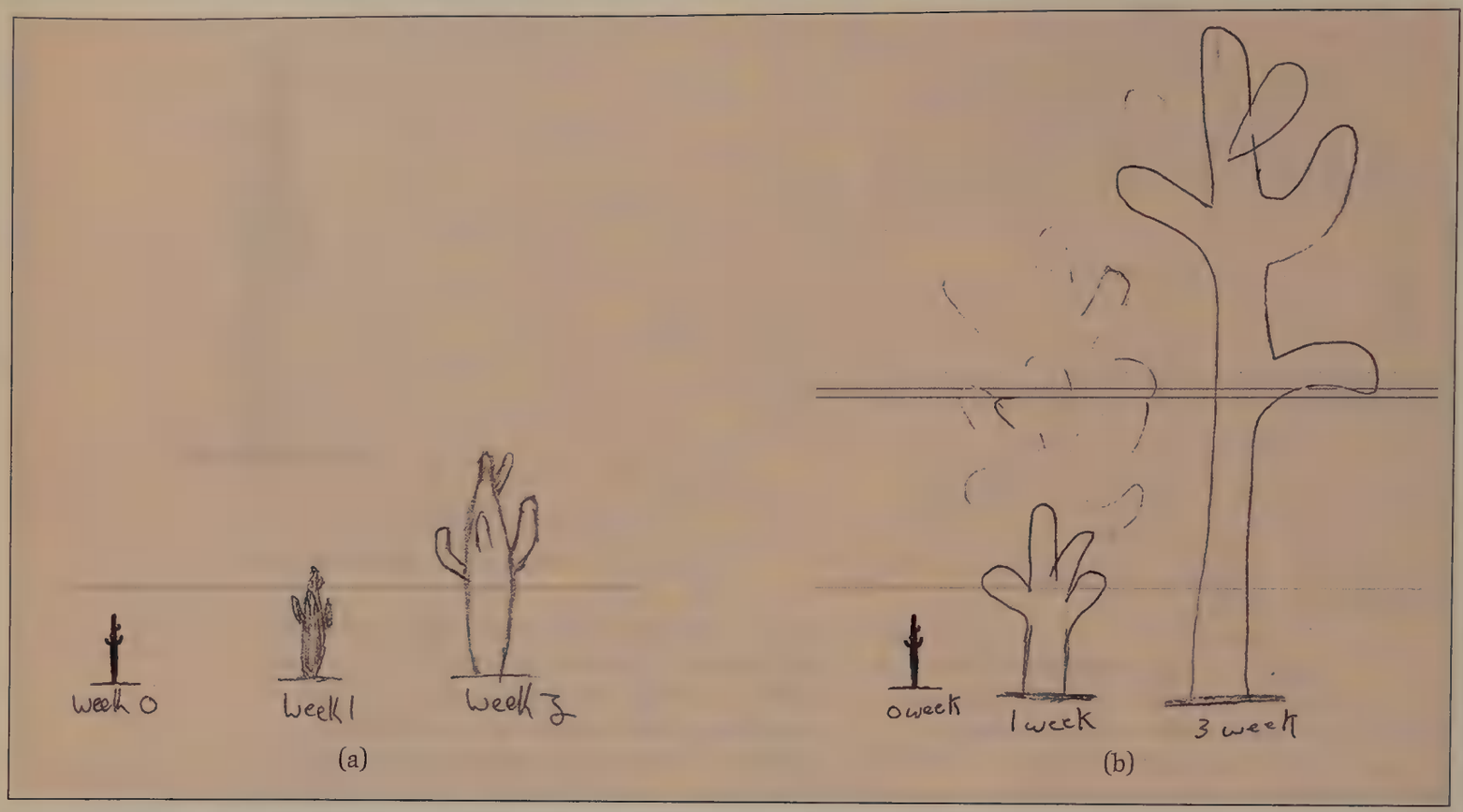


Fig. 4 Students draw the Jactus after one week and after three weeks; the work of Juan (**a**) and Paj (**b**) is shown.

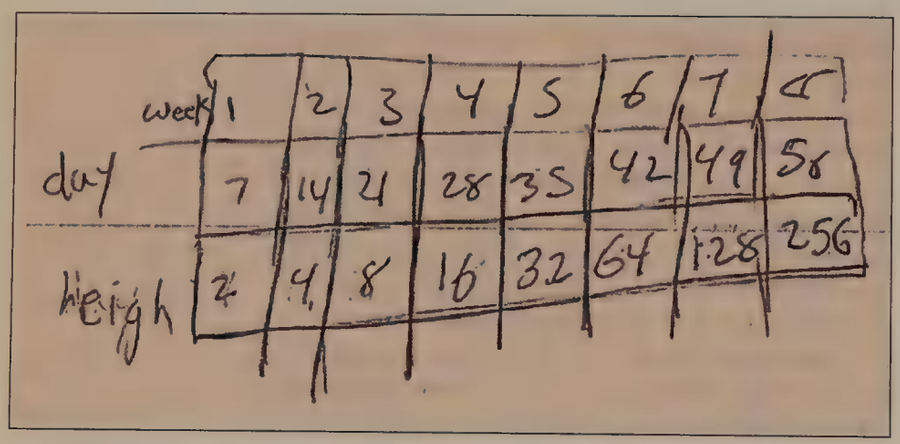


Fig. 5 Paj's table of height and time values represents the Jactus plant's growth.

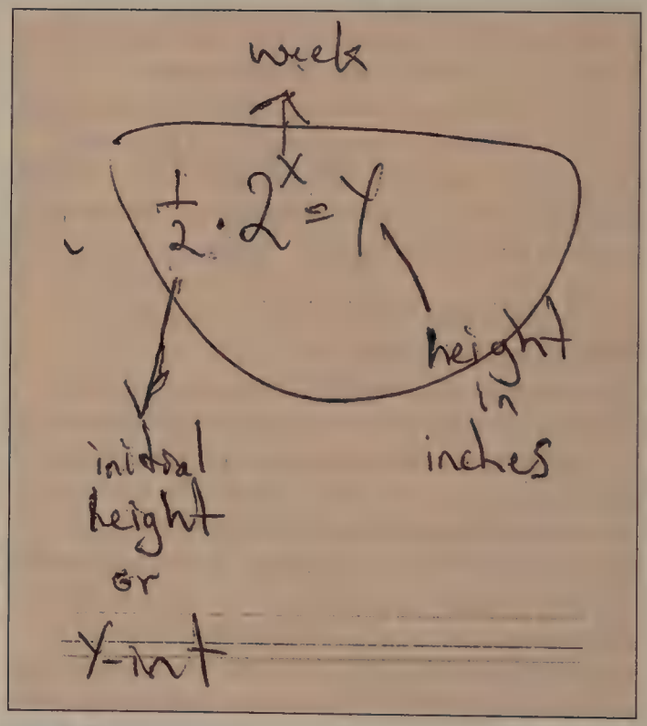


Fig. 6 Bonito's equation for the Jactus plant's growth uses an initial height of 1/2 inch.

or relationship is represented by each variable or parameter in their equations supports a more meaningful link between the situation and the models.

Once students are comfortable with the models that they have constructed, introducing data in different forms encourages students to revisit aspects of their models and rethink how to relate quantities in new ways (see principles 4 and 6). For instance, teachers could introduce tables of data with missing values or gaps between the weeks (see **fig. 7**). In this case, Bonito attempted to coordinate the increase in height with the increase in time (see **fig. 7a**), determining that the plant would grow four times as large in a two-week interval (notice that Bonito wrote "2" between week 3 and week 5 and then wrote "× 4" between 24 inches and 96 inches). However, he decided that the growth factor for three weeks would be 12 rather than 8 (as seen in Bonito's writing "× 12" between 96 inches at 5 weeks and 768 inches at 8 weeks). In contrast, Carter's work (see **fig. 7b**) shows his coordination of the growth in height for three weeks as represented by 2^3, or $2 \cdot 2 \cdot 2$. This correlation is evidenced by the fact that Carter wrote "$2 \cdot 2 \cdot 2$" between the height at week 10 and the height at week 13.

Teachers can encourage discussion and exploration of the relationships between height and time for different "chunks" of weeks to help students see how their exponential models hold, regardless of the value of Δx. They could also examine the students' models to see whether they would still hold for non–whole number exponents—for instance, by asking students to consider the plant's height at day 8 (rather than week 8). Once students can make calculations such as $3 \cdot 2^{(1+1/7)}$ to find the plant's height for a given day, they will begin to see the exponent x in $y = ab^x$ as representing any amount of time, rather than limited to a whole-number week value. The GeoGebra script can also support this conception; students can drag the base of the plant smoothly across the x-axis to observe the plant's height change continuously rather than in discrete chunks. By situating questions and explorations within the world of the Jactus plant and its quantities of height and time, teachers can help students develop meaningful models and representations of abstract concepts such as continuous exponential growth.

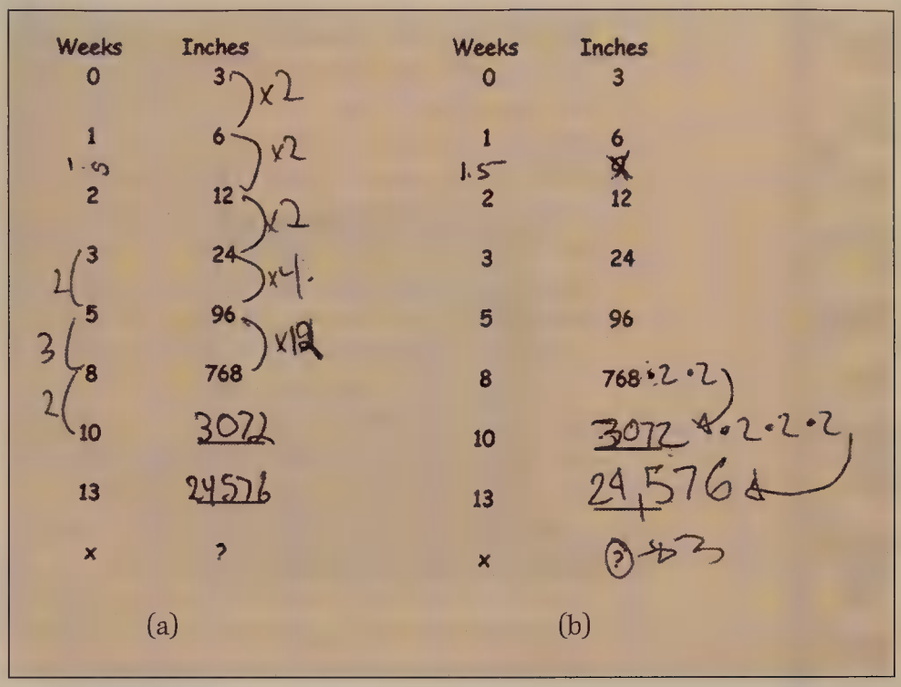


Fig. 7 A table of heights with nonuniform time increments shows Bonito's work (**a**) and Carter's work (**b**).

EXTENDING THE PRINCIPLES

Four Ways to Modify Tasks

The six principles described earlier focus primarily on engaging students in quantitative reasoning through patterns in modeling problems. The importance of modifying textbook exercises and classroom tasks in ways that support these six principles led us to modify our own classroom tasks in the following ways.

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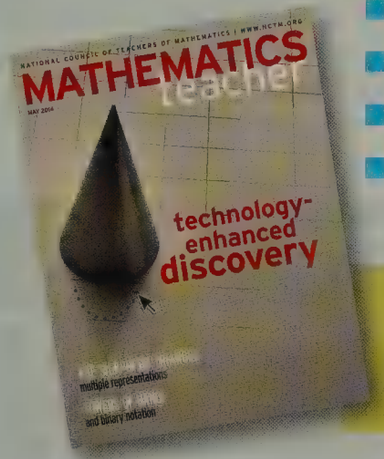

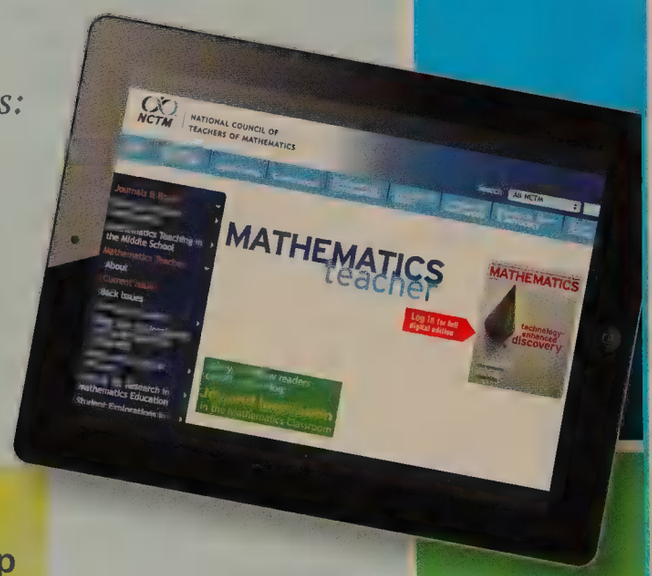

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1. Ensure that a task describes a situation in which the students themselves must identify quantities relevant to solving the problem rather than prescribing the quantities for them.
2. Create tasks in which the students must attend to the measures of the quantities in the problem as they determine relationships between those quantities. Make attention to the units important by using nonstandard measurements or asking the students to introduce their own measurements.
3. Create natural subparts to a task in which students must articulate their model for a situation and the quantities that constitute it. Doing so allows the students to reflect on the steps they took to solve the problem and to identify natural points at which to rethink their approach.
4. Introduce follow-up questions to tasks that create opportunities for the students to revise their models. These questions often arise from observing students struggle with a particular part of a problem.

Connections to Research

These two examples—the moving car and the growing plant—demonstrate how the principles we have proposed about quantitative reasoning in support of modeling can be enacted in the classroom. Beyond the evidence and experiences presented here, research has also identified quantitative reasoning as foundational to students' ability to think about functions, rate of change, generalization, and proof (Ellis 2009; Oehrtman, Carlson, and Thompson 2008; Weber 2012; Weber et al. 2012). This research suggests that the inclusion of modeling in textbooks and more recently in the Common Core State Standards for Mathematics requires teachers to think carefully about how to prepare and revise lessons to focus on modeling. The six principles we have proposed provide a means for teachers to think about modifying and adapting existing problems to support students' abilities to model through quantitative reasoning.

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ERIC WEBER, eric.weber@oregonstate.edu, is an assistant professor of mathematics education at Oregon State University in Corvallis. He is interested in how students think about functions of two variables and rate of change of two variable functions as well as how technology can support visualizing advanced mathematical ideas. **AMY ELLIS,** aellis1@wisc.edu, is an associate professor of mathematics education at the University of Wisconsin-Madison. She is interested in students' learning, particularly in the areas of algebraic reasoning, quantitative reasoning, and proof. **TORREY KULOW,** kulow@wisc.edu, is a doctoral student at the University of Wisconsin-Madison. She is interested in preservice and in-service teacher education, particularly in field placement. **ZEKIYE OZGUR,** zozgur@wisc.edu, is a doctoral student at the University of Wisconsin-Madison. She is interested in students' learning in the area of quantitative reasoning and proof.

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2016 Focus Issue

Teaching Mathematics Online

Today's students live in a world of computers, tablets, smartphones, and cloud computing. The pace of innovation continues to accelerate. How can mathematics teachers capitalize on these trends for the good of our students? The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, education researchers, and others to share their experiences in online mathematics education. Submissions that highlight how online technologies can support NCTM's Equity Principle are especially encouraged. In some cases, print alone will not adequately capture the total impact of the article; in such cases, having active links or other dynamic resources will be optimal.

All possible innovative uses of online resources cannot be enumerated here, but the following questions will get the conversation started.

OPPORTUNITY

How can online technologies—

- be leveraged for mathematics learning?
- provide for student collaboration across distances?
- improve professional development opportunities?
- affect the achievement gaps between different groups of students?

IMPLEMENTATION

How do you—

- remain connected to your students when teaching online?
- use specific online tools to help students learn?
- balance and complement traditional classroom formats with online technologies?
- compare the effectiveness of different methods for student learning?

These are just a few of our ideas; we want to know what you think. We particularly encourage teachers to share their own favorite online lessons.

Please submit manuscripts and Web materials at **mt.msubmit.net** by **May 1, 2015**. Be sure to enter the call's title (Teaching Mathematics Online) in the Department/Calls field. No author identification should appear in the text of the manuscript. Manuscript guidelines are available at www.nctm.org/publications/content.aspx?id=22602. If you have ideas related to this topic and wish to discuss them before sending a manuscript or need details about restrictions on online resources as part of submissions, please contact Tara Slesar (tslesar@nctm.org).

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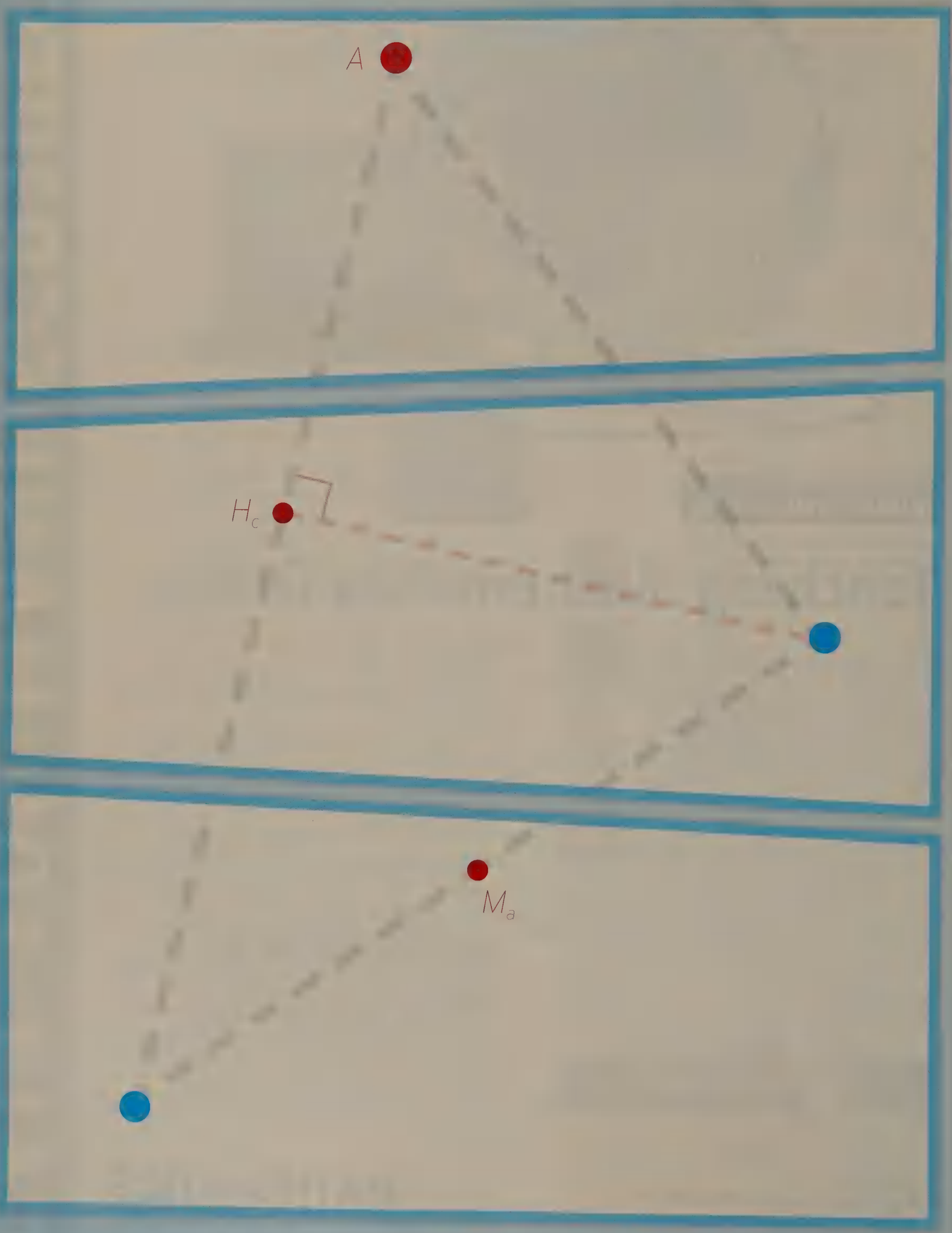
A
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TRIANGLES FROM THREE POINTS

Using technology to solve triangle construction problems, students apply their knowledge of points of concurrency, coordinate geometry, and transformational geometry.

Wayne Nirode

Geometry students need challenges. They need to apply what they already know to new contexts. As a result, I am always looking for groups of related problems or theorems to challenge my geometry students.

I came across one such group of problems when reading Jun's (2012) one-page abstract posted online for the 12th International Congress on Mathematics Education. Jun referenced Wernick's (1982) original work in which he posed 139 unique problems. In each problem, three points are chosen from the following set of sixteen points—the three vertices of a triangle, the four classic points of concurrency (i.e., centroid, circumcenter, incenter, and orthocenter), the three midpoints, the three perpendicular feet, and the three feet of the angle bisectors. The task is to locate, if possible, the three vertices of the triangle. For example, one possible version of the problem is to locate the other two vertices of a triangle given a vertex, the orthocenter, and the centroid. (For a complete list of Wernick's problems, go to http://dx.doi.org/10.2307/2691396.)

I implemented Wernick's problems in my instruction with my freshmen honors geometry students in two distinct ways. First, I gave students numeric coordinates for three points and asked them to find the coordinates of the vertices of the triangle. Second, I followed Jun's suggestion to use dynamic geometry software (DGS) and gave students problems to complete as a project using The Geometer's Sketchpad® (GSP). (Other versions of DGS also could be used, such as GeoGebra, which is free.)

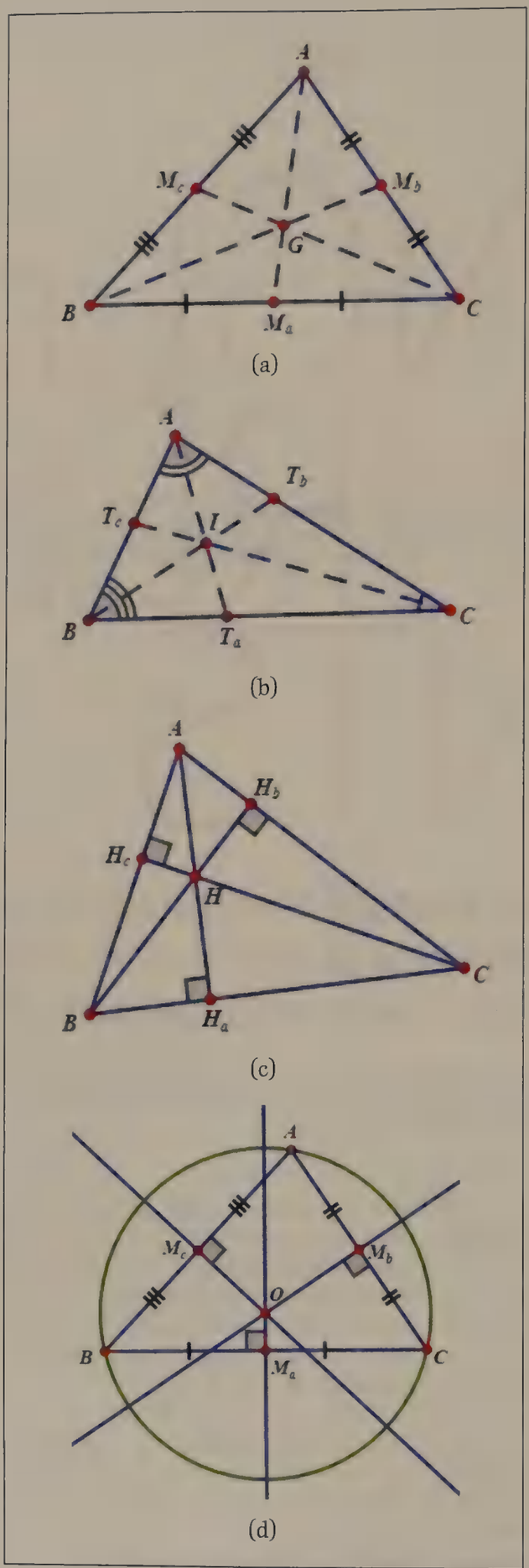

Fig. 1 The sixteen characteristic points of triangle *ABC* are shown. Intersecting segments locate the centroid *G* (**a**), the incenter *I* (**b**), the orthocenter *H* (**c**), and the circumcenter *O* (**d**).

AN ASSIGNMENT USING NUMERIC COORDINATES

During the first quarter of my honors geometry course, students completed extensive work with coordinate geometry and the points of concurrency. I emphasize coordinate geometry because it is an organic way for students to maintain their algebra skills while enrolled in a geometry course. Before this assignment, students were familiar with coordinate geometry; they could write equations of lines and circles, use the midpoint and distance formulas, and find intersection points. Students also had made several conjectures with points of concurrency during instructional activities that included paper folding and using GSP.

I initially gave students problems with numeric coordinates for three points based on Wernick's work. To solve these problems, students applied their knowledge of coordinate geometry. In addition, students used the definitions and the relationships among and between the points of concurrency as well as the segments that formed them. The following relationships were used:

1. The centroid divides the median in the ratio 2:1.
2. The circumcenter is equidistant from the vertices of the triangle.
3. The circumcenter, centroid, and orthocenter are collinear (i.e., they lie on the Euler line).
4. The centroid divides the Euler "segment" into a ratio of 2:1 (orthocenter to centroid:centroid to circumcenter)

Standard notation was used for the sixteen points with $\triangle ABC$ as the reference triangle:

- A, B, and C are the vertices of the triangle with a, b, and c the sides opposite these respective vertices.
- M_a, M_b, and M_c are midpoints of sides a, b, and c, respectively (see **fig. 1a**).
- G is the centroid (the intersection of the medians) (see **fig. 1a**).
- T_a, T_b, and T_c are feet of the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively (see **fig. 1b**).
- I is the incenter (the intersection of the angle bisectors) (see **fig. 1b**).
- H_a, H_b, and H_c are feet of the altitudes from vertices A, B, and C, respectively (see **fig. 1c**).
- H is the orthocenter (the intersection of the altitudes) (see **fig. 1c**).
- O is the circumcenter (the intersection of the perpendicular bisectors) (see **fig. 1d**).

When I introduced these problems to the class, we discussed different ways to approach them. For some problems, we used vectors instead of a

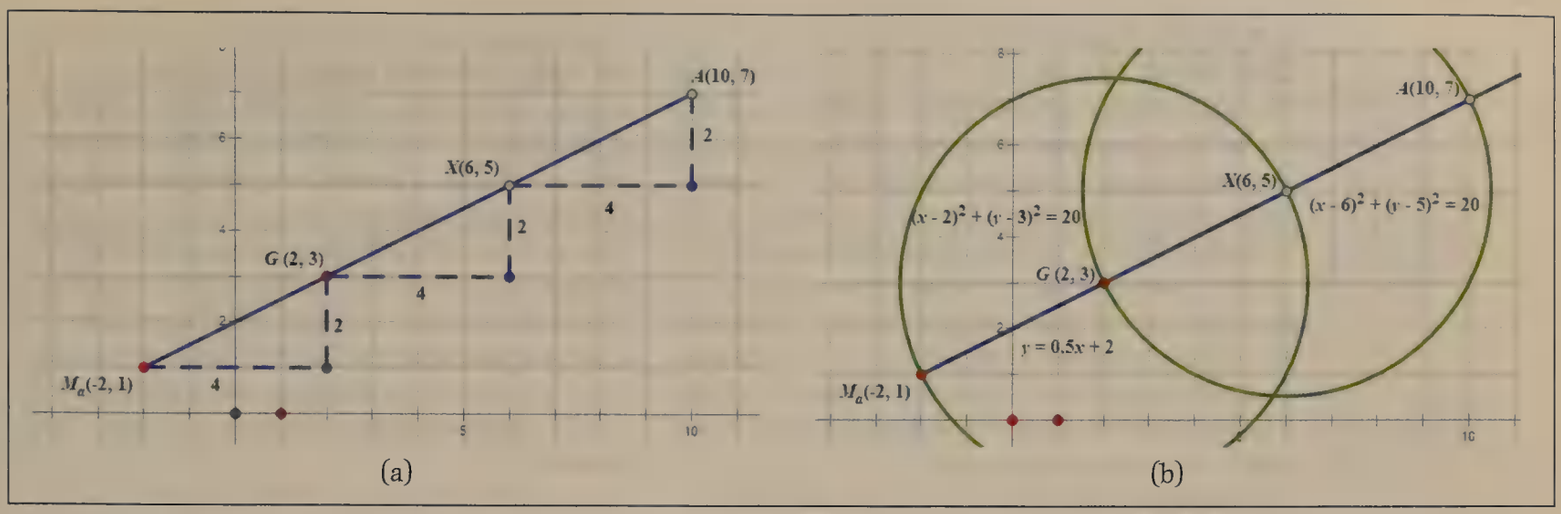

Fig. 2 Students can locate point A using a vector method to translate G by $2\,\overrightarrow{M_aG}$ (**a**). A graphical and algebraic approach can be used to locate A (**b**).

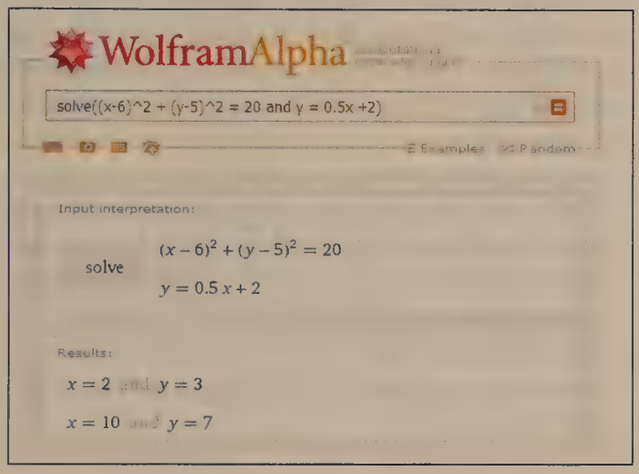

Fig. 3 WolframAlpha can be used to solve systems of equations.

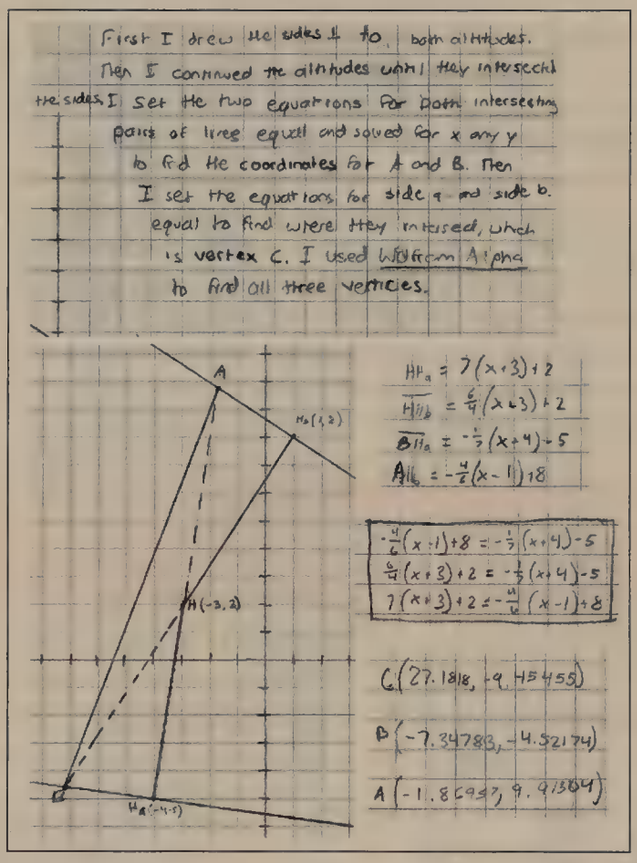
First I drew the sides & to both altitudes.
Then I continued the altitudes until they intersected the sides. I set the two equations for both intersecting pairs of lines equal and solved for x and y to find the coordinates for A and B. Then I set the equations for side a and side b equal to find where they intersect, which is vertex C. I used WolframAlpha to find all three vertices.
A
H(-3, 2)
C(27.1818, -9.45455)
B(-7.34783, -4.52174)
A(-1.85937, 9.97656)

Fig. 4 Matthew's solution shows his thinking about Wernick's problem 125.

more complicated algebraic method. For example, if the location of the centroid G (2, 3) and the midpoint M_a (–2, 1) are known, A can be located by translating G by the vector $2 \cdot \overrightarrow{M_aG}$ because of relationship 1. The magnitudes of the horizontal and vertical components of vector $\overrightarrow{M_aG}$ are 4 and 2, respectively, so G is translated to the right by 8 units and up by 4 units to locate A at (10, 7) (see **fig. 2a**).

On the other hand, an algebraic approach leads to a longer process involving equations associated with the following objects (see **fig. 2b**):

- Median M_aG, contained in $y = 0.5x + 2$
- Circle G with radius point M_a, represented by $(x - 2)^2 + (y - 3)^2 = 20$
- The intersection point of the median and circle G, represented by X (6, 5)
- The equation of circle X with radius point G, represented by $(x - 6)^2 + (y - 5)^2 = 20$
- The intersection point of the median and circle X, represented by A (10, 7)

To reduce some of the tedious symbolic manipulation with paper and pencil, I showed students how to use wolframalpha.com, the computational knowledge engine, to solve systems of equations. For example, students can use WolframAlpha® to find the intersection point of $(x - 6)^2 + (y - 5)^2 = 20$ and $y = 0.5x + 2$ (see **fig. 3**). A calculator with a computer algebra system (CAS) could also be used.

As one homework exercise, my students worked on Wernick's problem 125. Students located the vertices of $\triangle ABC$ given its orthocenter, H (–3, 2); the perpendicular foot of side a, H_a (–4, –5); and the perpendicular foot of side b, H_b (1, 8). Matthew's solution to and his thinking about this problem are shown in **figure 4**.

A PROJECT USING DYNAMIC CONSTRUCTIONS

Up to this point, my students had made limited use of The Geometer's Sketchpad. Even though they had always enjoyed using GSP, I had become increasingly dissatisfied with the activities I was using because they rarely afforded students opportunities to think and reason; instead, they were "cookbook lessons—those that lead students through a series of procedural steps in a recipe-like fashion" (Harper and Edwards 2011, p. 180). This year, I decided to use activities that did not give specific steps, thus requiring students to do more thinking. In using GSP in this way, I have found that my students often can explain to me what they want to do mathematically; they just need me to explain to them how to do it with GSP.

To begin the GSP project, I randomly assigned seven unique problems to each student and required that each student solve four of the assigned problems or explain why he or she could not. Although no students had the same problem, each knew what problems the others had, and I encouraged them to collaborate on similar problems. I also encouraged students to ask me for help. When students came to me for help, they would show me on the computer what they had tried so far, and I would offer suggestions for finding a solution. Students had a month to work on these problems outside class.

For each problem, students completed a GSP page showing the step-by-step process that they used to solve the problem. The diagrams had to be dynamic—that is, when points are dragged, the construction must still "work," although in some cases the triangle will no longer exist (e.g., two vertices and the incenter cannot be collinear). In addition to the skills used to solve the numeric coordinate problems, students needed to apply their knowledge of dilations, translations, reflections, and rotations to solve these with GSP. **Figure 5** shows a static version of Erin's solution to Wernick's problem 75: circumcenter, O, perpendicular foot of side a, H_a, and orthocenter, H. (For a dynamic version of Erin's solution, go to **www.nctm.org/mt057**.)

After completing the GSP project, students wrote a reflection. Students reported spending approximately five hours on the project. Most wrote that they enjoyed the project even though it was frustrating and challenging at times. Students seemed to like the fact that this assignment afforded them opportunities to explore and to think instead of working within the confines of a closed-ended task. Comments from four students are given here:

- "I thought this assignment was interesting, but very frustrating . . . I found it very surprising to see the different levels of difficulty from problem to problem."
- " . . . I thought this was also fun since, I think, playing around and trying different things in Sketchpad is fun."
- "From this project, I think, I learned that geometry is pretty cool in the way that most things are dependent on each other."
- "The project actually gave me some self-confidence. I would get so happy when I figured out how to do one and it would work."

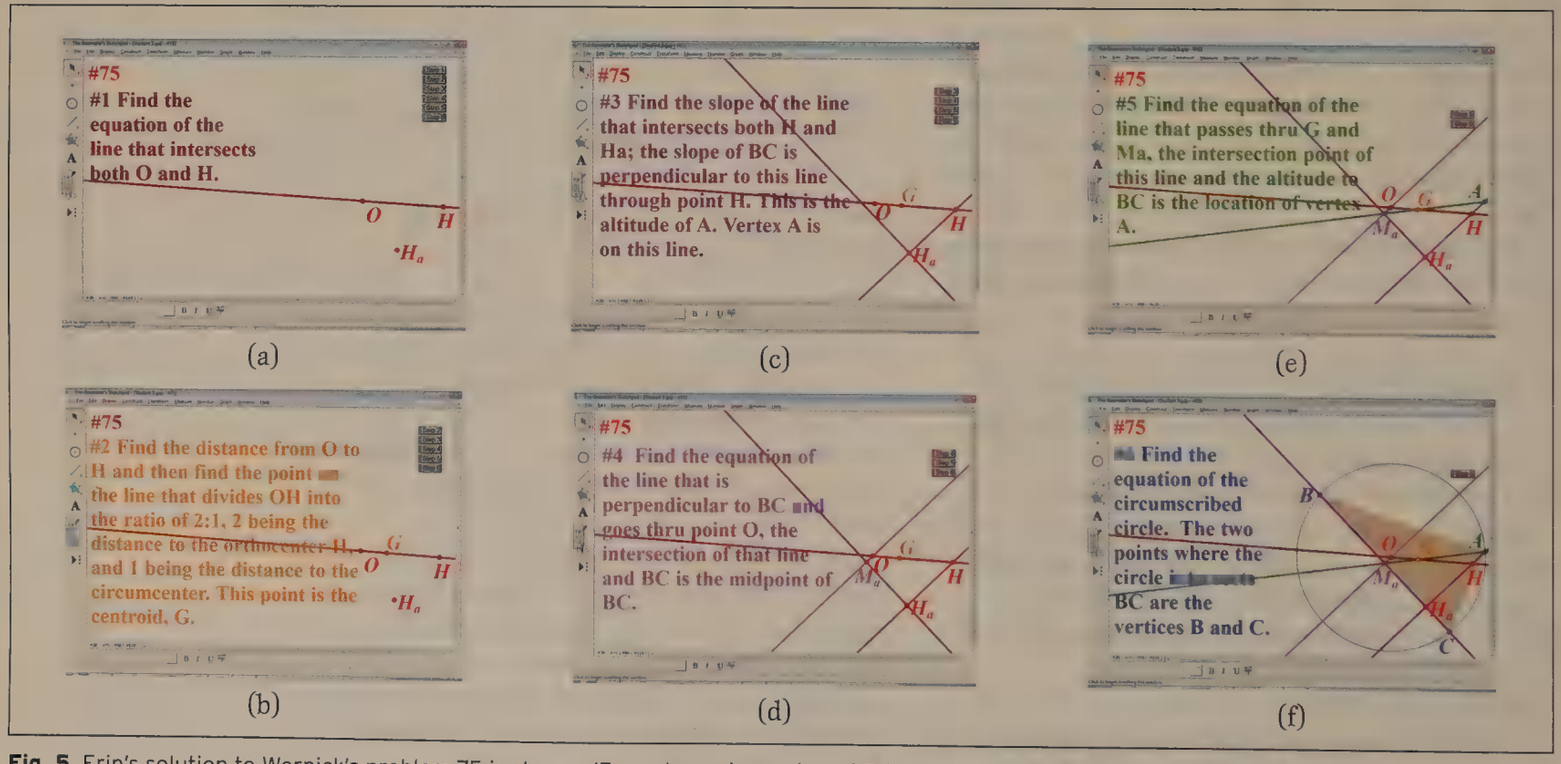
#75
#1 Find the equation of the line that intersects both O and H.
(a)
#75
#3 Find the slope of the line that intersects both H and Ha; the slope of BC is perpendicular to this line through point H. This is the altitude of A. Vertex A is on this line.
(c)
#75
#5 Find the equation of the line that passes thru G and Ma, the intersection point of this line and the altitude to BC is the location of vertex A.
(e)
#75
#2 Find the distance from O to H and then find the point on the line that divides OH into the ratio of 2:1, 2 being the distance to the orthocenter H, and 1 being the distance to the circumcenter. This point is the centroid, G.
(b)
#75
#4 Find the equation of the line that is perpendicular to BC and goes thru point O, the intersection of that line and BC is the midpoint of BC.
(d)
#75
Find the equation of the circumscribed circle. The two points where the circle intersects BC are the vertices B and C.
(f)

Fig. 5 Erin's solution to Wernick's problem 75 is shown. (For a dynamic version of Erin's solution, go to **www.nctm.org/mt057.**)

Sifting through Wernick's 139 problems and wondering whether the teacher or the students have the required knowledge to solve them can be overwhelming for a teacher (as it was for me). So I compiled a list of problems that have solutions accessible to students in a "standard" high school geometry course using my students' solutions, my solutions, and Sprecht's (2009) solutions (although in German) with diagrams as a guide (see **fig. 6**). Between Wernick's original work and Meyers's update (1996), these two authors identified 70 solvable problems out of 139 that did not have *locus solutions* (e.g., the problem given A, B, and O has a locus solution because O must lie on the perpendicular bisector of AB; then C can be anywhere on the circumscribed circle) or were not *redundant* (e.g., the problem given A, B, and M_c is redundant because knowing A and B determines the location of M_c; then C can be located anywhere that is not collinear with the three given points). **Figure 6** shows 44 of the 70 solvable problems with solutions that are accessible to high school students using basic definitions and the four conjectures stated previously.

Wernick's original list of 139 problems also can be used as a discussion starter among mathematics teachers about whether a teacher should give students problems that the students (and sometimes the teacher) may not know how to solve. The complexity of the task is increased when we do not know whether the solution is possible or, even if it is possible, whether we have the knowledge needed to solve it. Typically, students are asked to do only problems that can be solved—in fact, mostly they are asked to do problems that the teacher can solve and quickly help with if they struggle. I used the complete list, so the roles of teacher and student were more equal because I could offer suggestions; however, I was not certain that they would work. In discussing this dilemma with students after they had completed their projects, many found it unsettling to work on problems in this manner; they would frequently second-guess themselves because they were not sure whether they had exhausted all possibilities when trying to locate the vertices or whether they had overlooked something.

Although I am going to use these problems again next year, I plan to make some changes. Concerning the GSP project, students will be required to turn in a rough draft of one completed problem so that I can provide feedback before the final copy is submitted. I want to ensure that students use appropriate vocabulary when describing the steps that they undertook with GSP. For example, Erin

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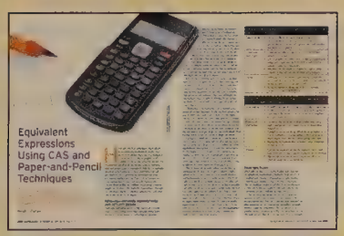
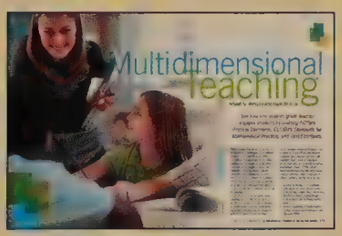
Linking research and practice has long been an NCTM strategic directive. NCTM's Research Committee has now advanced that goal with the Linking Research and Practice Outstanding Publication Award. Based on criteria ranging from timeliness to applicability, the annual award is given to a research-based article in one of the NCTM school journals. The 2013–2014 volume-year recipients are the following:



MATHEMATICS TEACHER
Nicole L. Fonger, "Equivalent Expressions Using CAS and Paper-and-Pencil Techniques" (May 2014, pp. 688–93).



MATHEMATICS TEACHING IN THE MIDDLE SCHOOL
Robert Q. Berry III and Mark W. Ellis, "Multidimensional Teaching" (October 2013, pp. 172–78).

TEACHING CHILDREN MATHEMATICS
Higinio Dominguez and Melissa Adams, "Más o Menos: Exploring Estimation in a Bilingual Classroom" (August 2013, pp. 36–41).

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NCTM congratulates the recipients of this award, who will be acknowledged at the 2015 NCTM Research Conference and at the Annual Meeting and Exposition in Boston.

NCTM

2. A, B, M_a	16. A, O, H	33. A, M_b, H	54. A, H_b, I	85. M_a, M_b, H_a
4. A, B, G	17. A, O, T_a	34. A, M_b, T_a	61. A, T_b, I	92. M_a, G, H_b
7. A, B, H	18. A, O, T_b	36. A, M_b, T_c	62. O, M_a, M_b	93. M_a, G, H
8. A, B, T_a	20. A, M_a, M_b	37. A, M_b, I	63. O, M_a, G	97. M_a, H_a, H_b
10. A, B, I	23. A, M_a, H_b	39. A, G, H_b	65. O, M_a, H_b	103. M_a, H_b, H
11. A, O, M_a	24. A, M_a, H	40. A, G, H	66. O, M_a, H	105. M_a, H_b, T_b
13. A, O, G	25. A, M_a, T_a	44. A, H_a, H_b	70. O, G, H_a	116. G, H_a, H
14. A, O, H_a	28. A, M_b, M_c	49. A, H_b, H_c	75. O, H_a, H	125. H_a, H_b, H
15. A, O, H_b	29. A, M_b, G	51. A, H_b, T_a	84. M_a, M_b, G	

Fig. 6 This chart shows problems from Wernick's original 139 problems that are accessible to high school students.

could have used more precise vocabulary when referring to a particular line (e.g., she does not explicitly refer to the Euler line but instead refers to it as the line through the circumcenter and the orthocenter). Also, a few students did not make the three points independent; instead, they first created the triangle and then their points. This construction meant that the three points were now dependent points and moved only when a vertex of the triangle was dragged.

Even though points of concurrency are not always included in the high school geometry curriculum and hardly mentioned in the Common Core State Standards for Mathematics, this set of problems embodies many of the Standards for Mathematical Practice (CCSSI 2010). Students engage in problem solving, making connections, and communicating mathematics. Students must reason and plan a solution for these nonroutine problems by applying their knowledge of definitions, conjectures, and theorems of points of concurrency. If numeric coordinates are used, then students apply their knowledge of slope, midpoint, distance, writing equations of lines and circles, and finding intersection points of lines and circles.

When working on these problems, students have opportunities to strategically select appropriate tools, both with and without technology. A CAS allows students to focus on the problem-solving aspect instead of becoming overburdened with the computational component of these problems. When students use dynamic geometry software to solve these problems in general, they use their knowledge of dilations, translations, reflections, and rotations. Because these problems are related rather than isolated exercises, students can make connections among the problems to help them find solutions. Requiring students to provide written explanations encourages them to work on communicating their mathematical thinking.

With this set of related problems, students have many learning opportunities to extend and apply their thinking—something not always found in the typical exercises in a high school geometry course.

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WAYNE NIRODE, nirode-w@troy.k12.oh.us, teaches mathematics at Troy High School in Troy, Ohio. His interests include geometry, statistics, and technology.

more 4U

For a student solution using GeoGebra, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt057**. For Wernick's original table with Meyers's updates, go to **www.nctm.org/mt**.

Great Problems

"A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem"
— George Pólya (1957)

What makes a great problem?

The Editorial Panel of *Mathematics Teacher* wants you to share your great problems with other teachers. The Panel invites manuscripts that address the following questions:

- What is your favorite mathematical task? What makes it a great problem for students? What learning opportunities does this problem provide? What opportunities for student inquiry does this problem provide?
- What nonroutine mathematical tasks have you tried with students in grades 8–14 or in teacher education courses? How do such tasks connect to core school mathematics topics? How do these tasks open up opportunities for all students to make connections across mathematical ideas?
- Where do you find problems that drive great lessons? What specific multimedia resources or technologies could enrich teachers' ability to pose mathematical questions in a classroom? How have you integrated such resources into your classroom instruction?
- What are examples of lessons or units that you have designed from great problems?

We would welcome regular feature articles or those submitted to departments such as Back Page: My Favorite Lesson; Activities for Students; Media Clips; Mathematical Lens; Delving Deeper; or Reader Reflections.

We are especially interested in mathematical tasks that engage students, promote sense making, and generate rich discussions and, in so doing, support NCTM's Process Standards and the Common Core Standards for Mathematical Practice.

You may submit your completed manuscript for review by accessing **mt.msubmit.net**. Indicate that the manuscript is being submitted in response to the call Great Problems. Be sure to enter the call's title in the Department/Calls field. No author identification should appear in the text of the manuscript. Additional guidelines for preparation of manuscripts can be found at **www.nctm.org/publications/content.aspx?id=10452**.

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This month's theme is Math and the Internet.

In June 2012, about 2.4 billion people used the Internet—100 times more than the number of people who used it in June 1995. If the growth in Internet usage is linear (arithmetic), about how many people were added on average each year from 1995 to 2012?

1

In the August 1 problem, if growth is exponential (geometric), how many people were added to the Internet count in the year ending June 2012?

2

The world population in June 2012 was 7,017,846,922. Find the percentage of the population that used the Internet in June 2012 (see the August 1 problem).

3

The chart gives the approximate number of websites (in millions) at the end of each year. Find a regression function and use it to estimate the number of websites expected by the end of 2014.

Year	1996	1998	2000	2002	2006	2008	2011
Sites	1.2	3.2	8	15.6	46.8	70	366.8

4

Moore's law states that the number of transistors on integrated boards doubles every 18 to 24 months. Many futurists believe that the doubling in Moore's law is applicable to a variety of technical disciplines. Assume that at the beginning of 1996 there were 1.2 million websites. Use Moore's 24-month doubling time to estimate the number of websites by the end of 2011.

5

Repeat the problem for August 5 but use the 18-month doubling time to estimate the number of websites by the end of 2011.

6

The results from the problems for August 5 and August 6 surround the actual value as indicated in the table of the August 4 problem. Find, correct to the nearest tenth, the number of months in the doubling time that would result in the number of websites shown for 2011 in the table.

7

The population of the United States is about 316,500,000, and the world population is 7,106,000,000. If 1 of 13 people in the world are on Facebook® and 71.2% of Americans are on Facebook, what percentage of non-Americans are on Facebook?

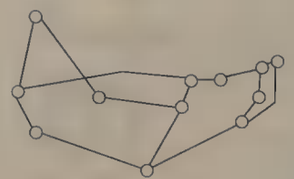
8

The table below gives the percentage of Twitter® users in each age group. Use the data to estimate the median and average ages of people who tweet, making any necessary reasonable assumptions.

Age	< 15	15–24	25–34	35–44	45–54	55–64	> 65
%	6	32	23	17	13	6	3

9

If adjacent vertices of the graph must be different colors, what is the minimum number of colors needed? (*Adjacent* means connected by an edge.)



10

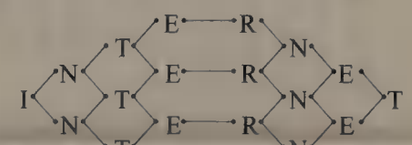
The number of annual Google™ searches is steadily increasing. Use the data in the table to find the average slope between consecutive pairs of points. Then compare that answer with the slope of the line joining the first and last points.

Year	2007	2008	2009	2010	2011	2012
No. of Searches/10^{11}	4.38	6.37	9.54	13.25	17.22	18.7

11

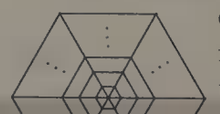
Regular pentagon *INTER* has an area of 125 tan(54°). Find the total length of the segments that take you from *I* to the final *T* as you spell out the word *INTERNET*.

How many different pathways will take you from the letter *I* on the left to the letter *T* on the right and pass through the letters in the word *INTERNET*?



Intern E.T. phoned home for some monetary help from his parents. He asked them for 23144 Asogi dollars to pay his rent. Asogi numbers are in base 6 (E.T. had only 3 fingers on each hand), and 7 Asogi dollars are equivalent to 1 American dollar. How many American dollars did E.T. pay for rent?

A spider web consists of 3 concurrent lines forming radial support for 20 concentric regular hexagons. The hexagons' vertices are 5 mm from their immediate neighbors. If the innermost hexagon has a side length of 3 mm, how many meters is the length of the entire web, including



If the roots of $x^2 - x + 2 = 0$ are r and t and the product of the roots is n, evaluate the expression $\text{Int}(e^r \cdot n \cdot e^t)$, where Int is the greatest integer function.

16

In 2006, there were no Twitter tweets. Five years later, in 2011, there were 200 million tweets per day. In 2013, the average number of tweets per day was 400 million. Find the annual percentage growth rate of tweets per day over the two-year period from 2011 to 2013.

17

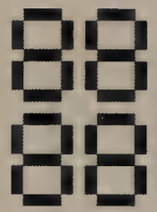
Mac computer monitors operate at 72 dpi (dots per inch). Since font size is measured in points (72 points per inch), Mac monitors are WYSIWYG (What You See Is What You Get). The image will be the same size in print as it is on the screen. Windows monitors have a resolution of 96 dpi. What percentage setting should you use on a Windows monitor so that a printed image is WYSIWYG?

18

Ppi stands for pixels per inch. A photograph printed at 300 ppi has a better resolution than one printed at 72 ppi. An image that is 576 pixels × 720 pixels will print at 72 ppi as an 8 × 10-inch picture. What will be the dimensions of the picture when the print setting is 300 ppi?

19

The TI-84 screen consists of horizontal rows of 96 pixels and vertical columns of 64 pixels. Each symbol on the home screen is made of pixels in a 5 × 7 array bordered by nonusable pixels to the right and below. What is the maximum number of symbols that can appear on the TI-84 screen at one time?



20

The TI-84 displays a ragged line when graphing $y = x$ on the ZStandard screen where $-10 \le x \le 10$ and $-10 \le y \le 10$. The **TRACE** option identifies (0, 0) and then (.21276596, .21276596). Use the information in the problem for August 20 and the fact that the rightmost and the bottommost pixels are not used on the graphics screen to explain how these coordinates are determined.

21

A portion of the graph of $y = x$ displayed on a TI-84 in ZStandard is shown. The ragged display shows some single pixels and some double pixels (side by side). Use the information in the problems for August 20 and 21 to determine how many of each type will appear on the full screen.

22

Monitors that display color pictures assign values to each pixel for red (R), green (G), and blue (B) shades. A number from 0 to 255 is selected for each of the 3 primary colors for every pixel. How many different color combinations are possible?

23

Disks in a tower of Hanoi are identified from top to bottom as 1, 2, 3, and so on. A computer can solve the puzzle by listing each disk moved, in order. For example, a 3-disk puzzle is solved as 1-2-1-3-1-2-1. If it takes a computer 1 second to display the 10-disc solution, how long would it take to display the solution to a 30-disc tower? (Hint: For the n-disc tower, $2^n - 1$ moves are required.)

24

The goal of the Jumping Coins puzzle is to move each coin to the opposite side. Two moves are possible: slide a coin one space or jump it over another coin. If it takes a computer 1 second to display the solution for the 30-coin puzzle, how long will it take to display the solution for the 300-coin puzzle?

25

Find three positive integers a, b, and c such that

$$a + b + c = 1000$$

and

$$a^2 + b^2 = c^2.$$

26

Compute the square root of the base-5 number 243331_5 and leave your answer in base 5.

27

To multiply two numbers, a computer can halve one number, ignoring any remainder, and double the other. It can repeat this process until one number becomes 1. Then adding all the doubled numbers whose associated half-number is odd gives a sum equal to the product of the original numbers. How is this process related to base 2?

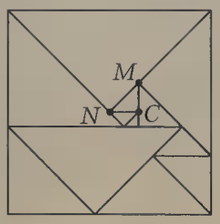
28

Determine a, b, and c if the number at the end of each row is the sum of the numbers in the row. Find a rule that relates the numbers at the bottom of each column to the entries in the column.

b	c	a	c	54
a	c	b	a	50
b	b	c	a	44
20	44	16	34	

29

It is possible to fill a square with right isosceles triangles of different sizes, as in the diagram. If $CM = CN = 2$, find the sum of the lengths of the 12 hypotenuses.



30

How many distinguishable arrangements of the letters

$$INTERNET$$

are there? Bonus: Do any of those arrangements form an English word?

31

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SOLUTIONS to calendar

1. 139,764,706. In the 17 years from June 1995 to June 2012, the number of users of the Internet grew to 2.4×10^9 from 2.4×10^7. So the average increase is $2.4(10^9 - 10^7)/17 = 139{,}764{,}705.882 \approx 139{,}764{,}706$. Alternatively, use the formula for an arithmetic sequence, $a_n = a_1 + (n - 1)d$, to solve for d. (Source of statistics in problem: http://www.internetworldstats.com/stats.htm)

2. 569,523,394. In an exponential growth rate model, $a_n = a_1 \cdot r^{n-1}$. Therefore, $2.4 \times 10^9 = 2.4 \times 10^7 \cdot r^{18-1}$. It follows that

$$r^{17} = \frac{2.4 \cdot 10^9}{2.4 \cdot 10^7} = 100$$

$$\text{so } r = \sqrt[17]{100} \approx 1.311134.$$

The number of users in June 2011 would have been $2.4 \times 10^9/1.3113393742 \approx 1{,}830{,}476{,}606$. The number of new users of the Internet from June 2011 to June 2012 $= 2{,}400{,}000{,}000 - 1{,}830{,}476{,}606 = 569{,}523{,}394$.

3. 34%. Divide the number of users by the total population: $2.4 \times 10^9/7{,}017{,}846{,}922 = 0.34199$. (Source of statistics in problem: http://www.internetworldstats.com/stats.htm)

4. 851 million. If $x = 0$ is the year 1996, then 1998 can be represented by $x = 2$. We are looking for the number of websites when $x = 18$. A scatter plot of the data suggests exponential growth. Regression gives $f(x) = 1.526 \cdot 1.421^x$, so $f(18) = 851$. (Source of statistics in problem: http://www.statisticbrain.com/total-number-of-websites/)

5. 217.2 million sites. Apply Moore's law, which states that doubling occurs every 2 years. We are asked to apply this law to find the number of websites after 15 years. With x as the number of doubling periods, use $f(x) = 1.2 \cdot 2^x$. So when $x = 15/2 = 7.5$, we find that $f(7.5) = 217.2$ million.

6. 1228.8 million sites. Since 18 months is 1.5 years, we can apply the formula $f(x) = 1.2 \cdot 2^x$ and substitute $x = 15/1.5 = 10$ to get $f(10) = 1228.8$. Notice that a small variation in doubling time makes for a big variation in the predicted number of sites.

7. 21.8 months. Let m = the number of months needed to reach 366.8 million websites. Then $m/12$ is the number of years, and $15/(m/12) = 180/m$ is the number of doubling periods in 15 years. Solve the equation $1.2 \cdot 2^{180/m} = 366.8 \rightarrow 2^{180/m} = 305.6667$. Take the log of each side to get

$$\frac{180}{m} = \frac{\log 305.6667}{\log 2}$$

$$\text{so } m = \frac{180 \log 2}{\log 305.6667} = 21.8.$$

8. 4.73%. The percentage of non-Americans on Facebook is the number of non-Americans on Facebook divided by the number of non-Americans in the world population. The number of non-Americans in the world is $7{,}106{,}000{,}000 - 316{,}500{,}000 = 6{,}789{,}500{,}000$. The number of people on Facebook is $7{,}106{,}000{,}000/13 = 546{,}615{,}385$, and the number of Americans on Facebook is $316{,}500{,}000 \cdot 0.712 = 225{,}348{,}000$. Therefore, the number of non-Americans on Facebook is $546{,}615{,}385 - 225{,}348{,}000 = 321{,}267{,}385$. The percentage of non-Americans on Facebook is $321{,}267{,}385/6{,}789{,}500{,}000 \approx 0.0473 = 4.73\%$. (Source of statistics in problem: http://www.worldometers.info/world-population/)

9. mean (average) = 32.6; median = 30.2. Finding the exact mean and median for grouped data sets is not possible. To estimate the mean, start by finding the midpoint of each interval and the product of the midpoint multiplied by the frequency of observations within the interval. Then calculate the sum of the products and divide by the total frequency. We adjusted the midpoints of the first and last intervals on the assumption that very few people younger than age 9 or older than age 79 tweet. So the mean is

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Department editors
Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

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Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, $22.95) and the 100 Problem Poster (stock number 13207, $9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.–Eds.

$[6(12) + 32(19.5) + 23(29.5) + 17(39.5) + 13(49.5) + 6(59.5) + 3(72)]/100 = 32.625$.

To estimate the median, notice that each age group contains the same span of 10 years, with the exception of the first and last groups. To find the median age of people who tweet, we need to find where the 50% mark falls. It falls in the age group 25–34 because 38% are below that interval, and 39% are above that interval. The interval 25–34 contains 23% of the data, and we would want 12% to fall below the median (because 12 + 38 = 50). We do not know exactly what number to use, but a reasonable approximation is to find a number that is 12/23 of the distance from 25 to 34, or $25 + (12/23) \cdot (34 - 25) \approx 29.7$. Statisticians might want to estimate the median for grouped data by applying the formula

$$\text{median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right),$$

where l is the lower bound of the interval containing the median, h is the size of the interval containing the median, f is the frequency of the median interval, n is the total frequency, and c is the cumulative frequency for all the intervals less than the median interval. The resulting estimate gives the median = $25 + (10/23)(100/2 - 38) \approx 30.2$. (Source of statistics in problem: https://www.syso mos.com/insidetwitter/)

10. 3 colors. The degree of a vertex is the number of vertices adjacent to it. Start by identifying all vertices with the highest degree (in this case, degree 3), selecting one of them (say, the bottom vertex), and coloring it blue. Use blue on other vertices with the same degree where possible, remembering that adjacent vertices may not be the same color. If no other vertex of the same degree can be colored blue, find all the vertices of the next lesser degree. Begin coloring these lesser-degree vertices blue and continue with the color blue as you move through the vertices in descending order (see **fig. a**). When no more vertices can be colored blue, return to the uncolored vertex of highest degree and color it red. Repeat the previous steps until no more vertices can be colored red. Then color the remaining vertex of highest degree yellow and follow the same algorithmic pattern. This is called the Welsh and Powell algorithm. (This coloring is not unique, but the graph cannot be colored in fewer than 3 colors.)

(a)

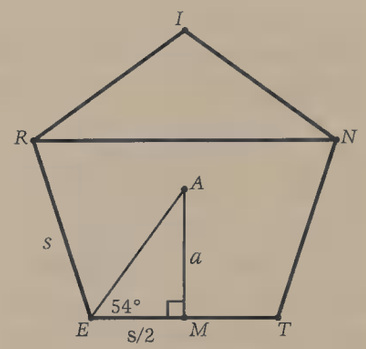
(b)

(c)

11. 2.872. The average of the slopes is the sum of the five slopes divided by 5. Each slope is the difference between successive numbers of searches because the successive years increase by 1. Therefore, the average is $(1.99 + 3.17 + 3.71 + 3.97 + 1.52)/5 = 14.36/5 = 2.872$. Since the x-coordinates of the points are evenly spaced, this result is the same as the slope determined by just the first and last points. The slope of the least squares line is 3.087.

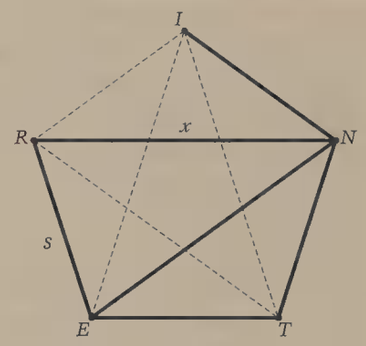
Here are some follow-up questions to explore: Why is the average of the slopes not the same as the slope of the line of least squared deviations? Find each subinterval of time when the growth rate was greater than the average rate over the entire interval. If you had to fit a model and make a prediction for 2013, would you use a linear or a nonlinear model? (Source of statistics in problem: http://www.statisticbrain.com/google-searches/)

12. $60 + 10\sqrt{5}$. The first step is to calculate the length of a side of a regular pentagon given its area. A formula for the area of a regular polygon is $A = ap/2$, where a is the length of the apothem and p is the perimeter. Let s be a side of the pentagon; then the perimeter is $5s$. Since each interior angle of a regular pentagon is 108°, the length of the apothem is $s\tan(54°)/2$. Thus, the area is $A = (s\tan(54°)/2)(5s)/2$, so $125\tan(54°) = 5s^2\tan(54°)/4 \rightarrow s^2 = 100 \rightarrow s = 10$ (see **fig. a**).



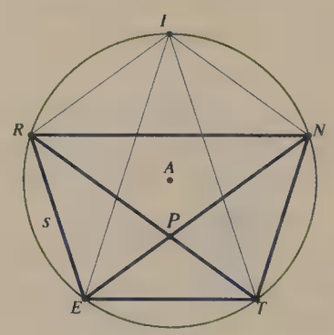
(a)

The segments that connect the letters in the word *INTERNET* comprise 5 sides of length s and 2 diagonals of length $x = s(1 + \sqrt{5})/2$ (see **fig. b**).



(b)

To determine the diagonal length RN, inscribe the pentagon in a circle but ignore vertex I (see **fig. c**).



(c)

Because $RNTE$ is a quadrilateral inscribed in a circle, we may apply Ptolemy's theorem: the product of the diagonals equals

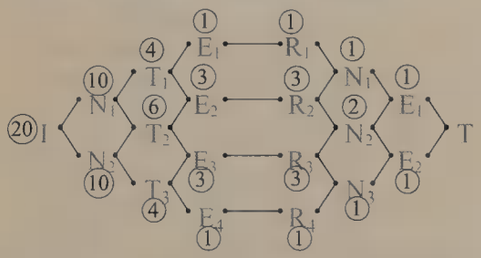
the sum of the products of both pairs of opposite sides. In our diagram, $RT \cdot NE = RE \cdot NE + RN \cdot ET$. Using side length $s = RE = ET = NT$ and diagonal $x = RN = RT = NE$, we have $x^2 = sx + s^2 \rightarrow x^2 - sx - s^2 = 0$. Apply the quadratic formula to solve for x in terms of s to get

$$x = \frac{s \pm s\sqrt{5}}{2} = s\left(\frac{1+\sqrt{5}}{2}\right).$$

The negative value was rejected. Thus, the total length is

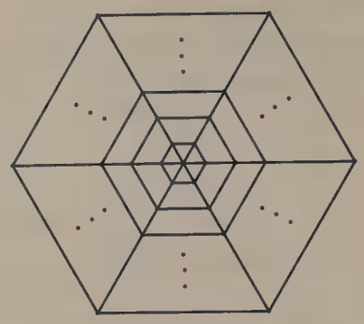
$$5s + 2x = 5(10) + 2(10)\left(\frac{1+\sqrt{5}}{2}\right) = 60 + 10\sqrt{5}.$$

13. 20. The circled numbers indicate the number of ways to get from one node (vertex) to the last letter, T. The numbers were found by working from right to left. From E_1 and from E_2, there is only one way to get to T. From N_1 and from N_3, there is only one way to get to T, but from N_2 there are two ways. Examine the results for the R column. From R_1 and R_4 there is one way, but from R_2 there are three ways, the sum of the two ways from N_1 and N_2. Each column is found in a similar way, by adding the numbers associated with each letter in the column to the right. Note the similarity to Pascal's triangle.



14. \$472. First convert 23,144 in base 6 to base 10: $2(1296) + 3(216) + 1(36) + 4(6) + 4(1) = 3304$. Since each American dollar equals 7 Asogi dollars, $3304/7 = 472$.

15. 6.648 m. The sides of the hexagons form an arithmetic sequence: 3, 8, 13, . . . , $3 + 19(5)$. The sum of these numbers can be found using $S = (3 + 98) \cdot (20/2) = 1010$. Since each hexagon has 6 sides, the total length in mm of the hexagons is 6060. Each hexagon is composed of 6 equilateral triangles; the diagonals of the hexagon are each $2(98) = 196$ mm, and $3(196) = 588$ mm. Thus, the total length of the entire web is $6060 + 588 = 6648$ mm, or 6.648 m.



16. 5. Reorder the expressions in parentheses to read $e^r \cdot e^t \cdot n$ and combine into ne^{r+t}. Because the sum of the roots is $r + t = 1$ and the product is $n = 2$, we get $ne^{r+t} = 2e^1 \approx 5.43656$. The greatest integer of a number is the greatest integer less than or equal to the number, so Int(5.43656) = 5.

17. 41.42%. Let r be the annual percentage growth rate over the two-year period. Then $200(1 + r)^2 = 400 \rightarrow r = \sqrt{2} - 1 \approx 0.4142 = 41.42\%$. (The percentage growth rate from 2006 cannot be calculated because division by 0 is not defined.)

18. 75%. We need to find what percent of 96 equals 72. To make the monitor display the same size as the print size, we need to set it at $72/96 = 0.75 = 75\%$. If the monitor is left at 96 dpi, the display size will be about 133% of the printed item.

19. 1.92 × 2.4 in. To find the new size, divide the number of pixels by the ppi. The image will be $576/300 = 1.92 \times 720/300 = 2.4$. The images here are examples at 72 and 300 ppi.

20. 128. Because 6 pixels, including a blank, are needed for each horizontal character and we have space for 96 pixels across, there is room for 16 symbols per row. The number of vertical pixels, 64, divided by the number of pixels plus empty space used vertically per character, 8, gives us 8 rows. Finally, we have $16(8) = 128$, the total number of characters per viewing screen.

21. Only 95 pixels are used horizontally, and only 63 pixels are used vertically. ZStandard sets the window so that $-10 \le x \le 10$ and $-10 \le y \le 10$. The **TRACE** option moves one pixel at a time left or right. There are 95 usable pixels, so there are 94 jumps from one pixel to the next. If we divide the length of the interval from −10 to 10 by 94, we get $20/94 = 0.212765957$, which rounds to 0.21276596, giving the horizontal step size that we see as we trace along a curve. The equation graphed is $y = x$, so the y-coordinate equals the x-coordinate.

22. 31 single pixels and 32 double pixels. There are 95 usable horizontal pixels on the graph screen. The ZStandard screen goes from −10 to 10 using 94 steps. Since $\Delta x = 20/94 \approx 0.2127659$, each step from one pixel to the next is about 0.2127659 units. Starting at −10 and adding Δx, we get 95 values {−10, −9.787234, −9.574468, −9.361702, −9.148936, −8.93617, . . .} that determine the screen column in which a pixel representing each point will be turned on. There are 63 usable vertical pixels and ZStandard goes from −10 to 10, so each step is $\Delta y = 20/62 \approx 0.3225806$. The 63 values {−10, −9.677419, . . .} determine the screen rows available for display.

If we use the calculator's right arrow key to follow the function $y = x$ from its leftmost point at (−10, −10), the next point will have x-coordinate $-10 + \Delta x = -9.787234$ and calculated y-coordinate −9.787234. However, the point can be represented by a pixel located only at a vertical position of −10 or $-10 + \Delta y = -9.677419$. Because −9.787234 is closer to −9.677419 than to −10, the second pixel will be located at (−9.787234, −9.677419). The calculator's **TRACE** feature displays a y-coordinate resulting from precise calculation, but the graphical output is restricted by the physical size of the pixels.

The vertical coordinate of the third point is −9.574468, but the third pixel can be located only at −9.677419 or −9.35483. Since the function's calculated y-value is closer to the lower option, the point is represented by a pixel at (−9.574468, −9.677419). Thus, the second and third pixels are in the same row, resulting in a double pixel.

For the most part, carefully looking at the screen shows that single and double pixels alternate except in the middle of the first and third quadrant. Two double pixels appear in a row midway through each quadrant (see the fig.). The first and last pixels are singles, so one-third of the pixels, or 31, are single and the remaining 95 – 31 = 64 pixels pair to form 32 double pixels.

23. 16,777,216 = 2^{24}. Since we have 3 colors and 256 values for each color, there will be 256(256)(256) = 16,777,216 possible color combinations, or $2^8(2^8)(2^8) = 2^{24}$.

24. 2^{20} sec. ≈ 12 days, 3 hr., 16 min., and 16 sec. The minimum number of moves to solve an n-disc tower of Hanoi puzzle is $2^n - 1$. Thus, the 10-disc puzzle requires $2^{10} - 1$ moves, and a 30-disc puzzle requires $2^{30} - 1$ moves, so the 30-disc puzzle uses about 2^{20} times more moves. Therefore, it will take about 2^{20} times longer. Instead of 1 sec., it will take about 2^{20} sec. To convert into days, use

$$2^{20}\,\text{sec.} \cdot \left(\frac{1\,\text{min.}}{60\,\text{sec.}}\right) \cdot \left(\frac{1\,\text{hr.}}{60\,\text{min.}}\right) \cdot \left(\frac{1\,\text{day}}{24\,\text{hr.}}\right)$$

$$\approx 12.1362963 \text{ days.}$$

Then (0.1362963 days) • (24 hr./day) ≈ 3.271111 hr., and (0.271111 hr.) • (60 min./hr.) ≈ 16.2666 min. Finally, (0.2666 min.) • (60 sec./min.) ≈ 16 sec. (For an analysis of the moves required to solve the tower of Hanoi problem using base 2, see "Twists on the Tower of Hanoi," by James Metz [*MT* May 2014, vol. 107, no. 9, pp. 712–15].)

25. 1 min., 34.375 sec. Begin by finding a formula for the least number of moves in the Jumping Coins puzzle. When there is one coin on each side, 3 moves are needed (1-2-1). With two coins, 8 moves are needed (2-3-4-2-1-3-4-1). With three coins on a side, 15 moves are required (4-3-2-4-5-6-3-2-1-4-5-6-2-1-6). Each number in the sequence 3, 8, 15 is 1 less than a perfect square. If we use n as the number of coins on one side, the minimum number of moves to arrange the coins is $(n + 1)^2 - 1 = n^2 + 2n$. The 30-coin solution requires $30^2 + 2(30) = 960$ moves and takes 1 sec. When $n = 300$, the number of moves is $300^2 + 2(300) = 90600$, taking 90600/960 = 94.375 sec. or more than 1.5 min. (For more about this game, see "Mathematizing 'Frogs': Heuristics, Proof, and Generalization in the Context of a Recreational Problem," by William Higginson [*MT* Oct. 1981, vol. 74, no. 7, pp. 505–15], and "Patterns Jumping out of a Simple Checker Puzzle," by Susan G. Staples [*MT* Nov. 2004, vol. 98, no. 4, pp. 224–27].)

26. (200, 375, 425). Use your knowledge of Pythagorean triples to list the first few: (3, 4, 5); (5, 12, 13); (8, 15, 17). Notice that the sums of the numbers in each of the three triples are 12, 30, and 40, respectively. Because 1000 is divisible by 40, multiply each number in the triple (8, 15, 17) by 25 to get (200, 375, 425). For an extra challenge, prove that this is the only triple whose sum is 1000.

27. 341_5. We can use any of the square-root algorithms that work in base 10 in base 5. For example, the strategy of divide and average can be applied. Take a guess at the square root of 243331_5—say, 123_5—and divide. So $243331_5 \div 123_5 \approx 1432_5$, and the average of the divisor and quotient is $(1432_5 + 123_5)/2 = 1030_5$. Now repeat the process by dividing by 1030_5. So $243331_5 \div 1030_5 \approx 230_5$, and the average is $(1030_5 + 230_5)/2 \approx 403_5$. Dividing by 403_5 results in a quotient of 324_5. Averaging 403_5 and 324_5 results in 341_5. Dividing 243331_5 by 341_5 results in 341_5, which is the square root of 243331_5. We can also check by converting the problem and solution into base 10. Students who are familiar with the digit-by-digit algorithm for square roots should try that as well. See page 46 for what that would look like in base 5.

Problem of the Month Retired

Mathematics Teacher is moving in a new direction and will no longer be soliciting reader and student solutions to a highlighted Problem of the Month in the Calendar. We wish to acknowledge the contributions of Seán Madden and Ricardo Diaz, who have been the editors of this department since its inception. We also wish to acknowledge readers who have responded to previously published problems, especially students who have risen to the challenge and shared their solutions:

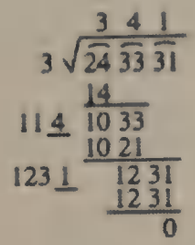
- Gayathri Ganesan and Jenson Lo, students of Frederick Deppe at Ithaca High School in Ithaca, New York
- Brandon Allin and Neal Mydonick, students of Michelle Haberstock at Parkland College in Yorkton, Saskatchewan, Canada
- Jack Stefanski, Ben Garrett, and John Lucas, students of Dick Smith at the University of Dubuque in Dubuque, Iowa
- Juliana Valencia and Andrea Beltran, students of Justine Aten and Lindsey Koch at Lincoln Park High School in Chicago
- Sal Culosi, a self-described mathematics enthusiast
- Augustine Mulai, a former exchange student of Seán Madden at Greeley West High School in Greeley, Colorado, who is currently pursuing higher education options in Freetown, Sierra Leone

more 4U

Although the Problem of the Month will no longer appear in the journal, the *MT* website features solutions to past problems (go to **www.nctm.org/mt**).

Solution 30

Triangle	*COP*	*CPN*	*NCM*	*MOL*	*KLG*	*KGF*	*FKE*	*EDI*	*IEL*	*PIJ*	*JNH*	*HMG*
Hypotenuse	$\sqrt{2}$	2	$2\sqrt{2}$	$3\sqrt{2}$	4	$4\sqrt{2}$	8	$6\sqrt{2}$	12	$8\sqrt{2}$	14	10



28. First, examine an example of the process—say, 19 • 25. For convenience, list this as an ordered pair (19, 25). Divide the first number by 2, ignoring the remainder, and multiply the second number by 2 to get (9, 50). Repeat this process to obtain (4, 100) → (2, 200) → (1, 400). Now add the numbers 25, 50, and 400 because the first numbers in those ordered pairs are odd. The sum is 475, which is the product of 19 by 25. Here is an explanation as to why it works: $19 \cdot 25 = (9\frac{1}{2})(50) = 9 \cdot 50 + 1 \cdot 25 = (4\frac{1}{2})(100) + 1 \cdot 25 = 4 \cdot 100 + 1 \cdot 50 + 1 \cdot 25 = (2)(200) + 0 \cdot 100 + 1 \cdot 50 + 1 \cdot 25 = 1 \cdot 400 + 0 \cdot 200 + 0 \cdot 100 + 1 \cdot 50 + 1 \cdot 25 = 475$. Notice that when the first number becomes a 1, all subsequent nonzero products correspond to the discarded remainders. To see the connection to base 2, write 19 as 10011_2. For another example, multiply 29 • 75 by converting 29 into base 2, as 11101_2, and then finding the sum $\mathbf{1}(1200) + \mathbf{1}(600) + \mathbf{1}(300) + \mathbf{0}(150) + \mathbf{1}(75) = 2175$.

MATHEMATICS teacher

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29. $a = 13$, $b = 7$, and $c = 17$. Double the first number + the second number – the third number. From the rows, we obtain three equations: $a + b + 2c = 54$; $2a + b + c = 50$; and $a + 2b + c = 44$. Adding these equations together results in $4(a + b + c) = 148$, or $a + b + c = 37$. Subtracting this result from each equation gives $c = 17$, $a = 13$, and $b = 7$, respectively. If we assume that the rows are linearly dependent, we can look for values x, y, and z such that x times the first row plus y times the second row plus z times the third row gives the values in the final row. We will obtain $x = 2$, $y = 1$, and $z = -1$, so the rule that generates the column results is double the first number + the second number – the third number.

30. $50 + 24\sqrt{2}$. Make a table listing the twelve triangles and calculate the hypotenuse length of each. Use the fact that in a right isosceles triangle the hypotenuse length is the leg length times $\sqrt{2}$.

Since $CN = 2$, we have $MN = 2\sqrt{2}$ and $NP = CP = \sqrt{2}$. Notice that $CO = OP = 1$, so $MO = 3$ and $ML = 3\sqrt{2}$.

The unknown length LG requires some work. Let LG be x, so $MG = 3\sqrt{2} + x$. It follows that

$$HG = \sqrt{2}MG = \sqrt{2}\left(3\sqrt{2} + x\right) = 6 + x\sqrt{2}.$$

Since $LG = x$, then $KG = x\sqrt{2} = GF$, so $HF = 6 + 2x\sqrt{2}$.

Separately, we can find the length JH in terms of x and then solve for x. Because $HN = HM + MN = MG + MN =$

$$\left(3\sqrt{2} + x\right) + 2\sqrt{2} = 5\sqrt{2} + x,$$

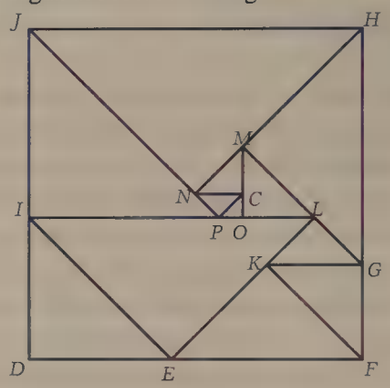
we have

$$JH = \sqrt{2}\left(5\sqrt{2} + x\right) = 10 + x\sqrt{2}.$$

Equating $JH = HF$ gives

$$6 + 2x\sqrt{2} = 10 + x\sqrt{2}$$
$$x\sqrt{2} = 4$$
$$x = 2\sqrt{2}.$$

Continue to find the hypotenuse of each triangle and then find the sum. The table displays the length of the hypotenuse for each right isosceles triangle.



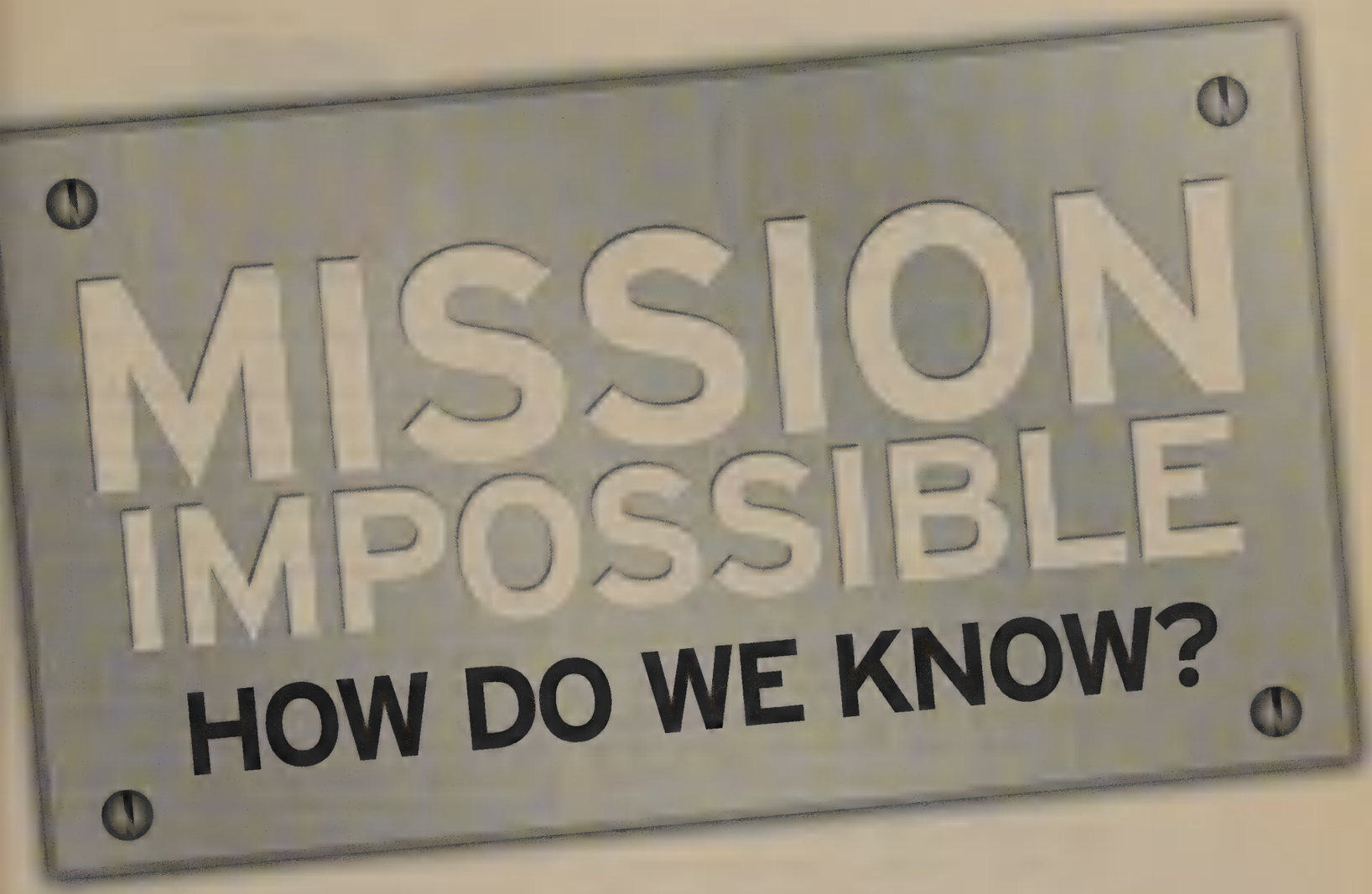
See **solution 30** table.

Alternate solution to find x = JH: Notice and prove that $NMLK$ is a rectangle, so $x = LG = LK = MN = 2\sqrt{2}$.

31. 5040 arrangements. The bonus answer is *renitent*. The number of distinct letter arrangements for the letters *INTERNET* is

$$\frac{8!}{2! \cdot 2! \cdot 2!} = 7! = 5040.$$

The only arrangement other than *INTERNET* that forms a word in English is *RENITENT*, an adjective that means resisting pressure or physical force.



Proofs of impossibility—very important in the history of mathematics—can provide additional opportunities for the development of reasoning, as recommended by the Common Core State Standards.

Julia Viro

Have you ever dreamed of being a hero who confronts the impossible? Did you envision yourself as Superman, flying through the air, or as Sherlock Holmes, deducing the truth by separating the impossible from the possible? Can you imagine achieving the impossible in mathematics? This article discusses the scenarios in which a mathematical task is impossible to accomplish as well as how to approach these scenarios in the classroom. The concept of proof is introduced as the primary tool to handle such situations.

Constructing viable arguments and reasoning abstractly is an essential part of the Common Core State Standards for Mathematics (CCSSI 2010). Indeed, the role of reasoning and proving in teaching mathematics can hardly be overestimated. Researchers emphasize the centrality of proof even in elementary school mathematics (Stylianides 2007) and much more so in a high school mathematics course (Alibert and Thomas 1991; Hanna 1991). Sadly, proof is regularly viewed as a kind of decoration or formal process rather than something applicable and pragmatic (Hanna and Jahnke 1993; Herbst 2002). On the contrary, it is important to present proving as a way to establish and explain

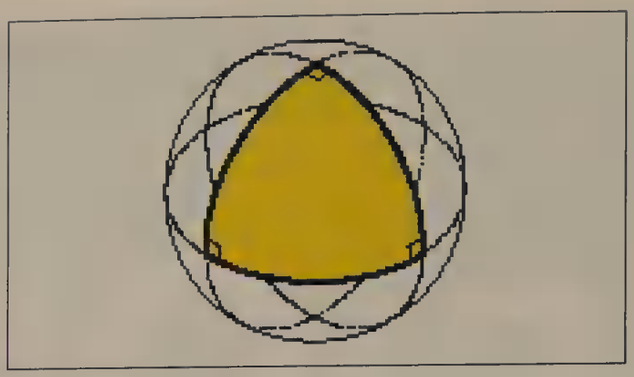


Fig. 1 A right equilateral triangle is formed on the sphere by arcs of three great circles.

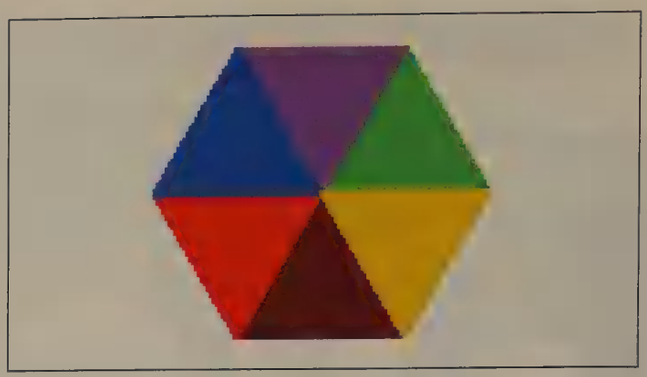


Fig. 2 A regular hexagon is made of six regular triangles.

mathematical truth (Dreyfus and Hadas 1996). Investigating impossibilities provides a great activity because the essence of logical argument presents itself there in pure form.

EASY, DIFFICULT, IMPOSSIBLE

"There is nothing impossible to him who will try"—these proud words of Alexander the Great sound like an excellent slogan. Once you have set a goal, work hard and achieve it. Sometimes achieving a goal is easy; sometimes it is difficult, but never say never! This common understanding of *impossible* is linked in our minds to our perception of difficulties: *impossible* refers to a difficulty that seems insuperable. And wouldn't it be glorious (although perhaps very challenging) to overcome such a difficulty?

In mathematics, however, we define *impossible* in a narrower sense—"unable to exist." If we want an object to have properties that contradict one another (meaning that they cannot happen together), we have to admit that such an object is impossible. In Euclidean geometry, a right equilateral triangle is impossible. Sometimes, what is impossible in one context becomes possible in another. If instead of the geometry of lines in the plane, we consider the geometry of great circles on the sphere, then a right equilateral triangle becomes possible (see **fig. 1**).

For an insider, mathematical impossibilities are no more mysterious than other mathematical facts. For students, impossibilities can take on the flavor of noble tasks for the chosen elite. This perception can be discouraging or attractive; the teacher can make the difference.

IMPOSSIBILITY SNEAKS INTO A CLASSROOM

Ms. Norris taught her geometry class how to construct an equilateral triangle using a straightedge and compass. Afterward, she suggested that the students think about how to construct a regular hexagon. It took some time, but her ninth graders remembered how to make a regular hexagon from six regular triangles (see **fig. 2**).

At this point, Ms. Norris mentioned that by using a straightedge and compass alone, it is possible to construct some other regular polygons—for example, a square and a pentagon—and impossible to construct a regular polygon with seven or nine sides.

The next day, Jason proudly presented to the class a beautiful regular polygon with nine sides: "I did it!" he announced. "It *is* a nice picture," responded Ms. Norris, "But you must have used some tools other than straightedge and compass." "I did not," insisted Jason, "Look, I purposely did not erase any of my constructions. I used constructions we learned and got an angle of 40°, then . . . ." "Wait, wait," interrupted Ms. Norris, "Actually, you have got an approximation, rather than an exact construction." Jason continued, "You may check with a compass—all the sides are the same length."

The unexpected discussion proved to be very productive although challenging. The difference between exact and approximate solutions was clarified, but, even realizing that the suggested approach had flaws, students were puzzled: Isn't it strange that angles of 30° and 60° can be constructed but that an angle of 40° cannot? How can we understand the difference?

History Helps

That students are puzzled by this kind of problem is not surprising. In fact, history tells us that mathematicians through the ages were challenged by several problems for which an expected solution does not exist. The construction of a regular 9-gon by a straightedge and compass is just one of them.

Even more famous is a problem of trisecting an angle—that is, developing an algorithm for dividing any angle into three congruent parts using only a straightedge and compass. Another famous problem is doubling a cube—that is, by a straightedge and compass, constructing an edge of a cube with a volume twice that of a given cube.

Probably the most famous of all is the problem of squaring the circle—that is, by a straightedge and compass only, constructing a square with an area equal to that of a given circle. The phrase "squaring a circle" has become an idiom to describe attempting a task that is truly impossible.

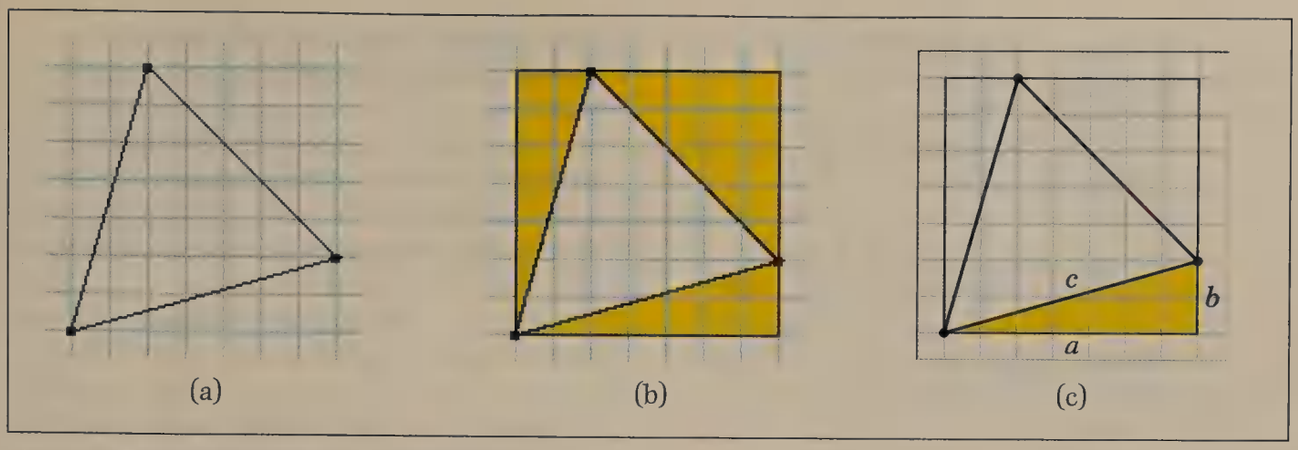


Fig. 3 Examining areas leads to a contradiction.

Since the time of the Ancient Greeks, this legendary trio of problems—squaring a circle, doubling a cube, and trisecting an angle—challenged generations of mathematicians and stimulated the development of mathematics (Eves 1990). Only in relatively recent times (in the nineteenth century) was it proved that these problems cannot be solved.

Bringing the dramatic story of these classical problems to the classroom would enrich students' perception of mathematics as a vivid, developing subject. The proofs of the impossibility of these constructions, however, are based on the quite sophisticated ideas of modern mathematics, including Galois theory and algebraic number theory. So discussing these topics with high school students is a challenge but not impossible. Some attempts of rather elementary exposition (including proofs) of the impossibility of doubling a cube and trisecting an angle are reported (Lützen 2010 and 2011).

Are any impossibilities elementary enough to discuss in the high school mathematics classroom?

ADVENTURES WITH EQUILATERAL TRIANGLES

Mr. Higgins distributes sheets of graph paper to the tenth graders in his honors geometry class. Today's problem sounds simple:

> Draw an equilateral triangle that has its vertices on the grid points of the graph paper.

Graph paper is always inviting for drawing. Most students apply their favorite strategy (trial and error), and pretty soon Ryan produces the picture shown in **figure 3a**. He claims that he has finished. "Not so fast," says Valerie, "Your triangle is only isosceles, not equilateral!" Mr. Higgins says: "That's correct. But why?" "Well," starts Valerie, "Suppose that the distance between lines on the graph paper is 1 unit. Then two sides of Ryan's triangle have length $\sqrt{53}$, and the third one has length $\sqrt{50}$." Ryan has to agree: "The triangle is isosceles but not equilateral!"

The class gets back to the race; everyone wants to catch this equilateral triangle. Other attempts are made, but none is successful. The triangle is stubborn. When two congruent sides are drawn, the third refuses to be of the same length. Students start to say, "It doesn't work. We can't do this." This is a crucial moment: No one in the class can find a solution, but could it be that no one at all can solve the problem?

When Proof Is the Only Way to Find an Answer

This is the most difficult part of the lesson. Mr. Higgins suggests assuming that the desired triangle has been already constructed and then exploring its properties. Further reasoning flows naturally. Here is where the graph paper really helps.

Our presumed equilateral triangle is boxed in by a rectangle. The area of this rectangle is an integer number of square units. The area of our triangle is the area of the rectangle less the area of three surrounding right triangles (see **fig. 3b**).

Each of these three right triangles has legs of integer lengths; so has half the area of a rectangle with integer sides. Each triangle's area is, therefore, an integer or an integer divided by 2. So the area of our "equilateral" triangle must be an integer or an integer divided by 2; in any case, its area is a rational number. Mr. Higgins emphasizes the qualitative nature of this statement and leads the class further: "Do we remember how to calculate the area of an equilateral triangle with side c?" Of course, that was in a test last week: "The area is $c^2\sqrt{3}/4$." But again, using properties of the graph paper and the Pythagorean theorem, we see that c^2, as the sum of squares of two integers, is an integer (see **fig. 3c**).

And so, because we know that $\sqrt{3}$ is irrational, we see that $c^2\sqrt{3}/4$ is also an irrational number.

Ryan exclaims: "But . . . wait a minute! We just calculated that this area is rational, and now you say that it is irrational! This is crazy!" The whole class agrees that the same number cannot be both rational and irrational; there exists a contradiction. Mr. Higgins explains: "Logically, this contradiction

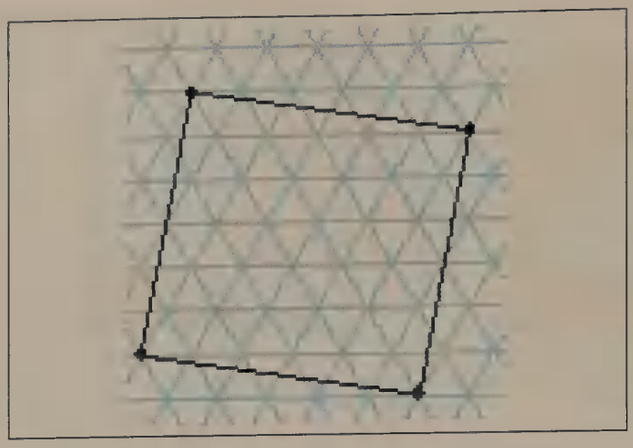


Fig. 4 Is it possible or not to construct a square on a triangular grid?

means that an erroneous assumption was made. Our only assumption was the existence of an equilateral triangle with its vertices placed on the grid points of graph paper. Therefore, such a triangle doesn't exist, and nobody in my honors geometry class can construct such a triangle. Nobody in the world can construct such a triangle!"

More in Store

This problem, like every good problem, leads to further discussions and generalizations. It follows immediately that any triangle that has an irrational area cannot be placed with its vertices on the grid points of a square grid. The same is true for any polygon with an irrational area. A dual problem for the given one can be formulated: Is it possible to place a square with its vertices on the grid points of a regular triangular grid? (See **fig. 4**.)

Going beyond the boundaries set by the problem and exploring the neighboring mathematical landscape is one of the real joys of mathematics.

WHEN INEQUALITY DISAPPEARS

Here is another example, from algebra rather than geometry and of a somewhat different nature. When students are introduced to complex numbers, they learn that these numbers can be added, subtracted, multiplied, and divided just like real numbers and that all the usual properties of these operations still hold true. Moreover, when we consider real numbers as complex numbers with imaginary part zero, then complex addition, multiplication, and all the operations give the same result as the usual addition, multiplication, and so on of real numbers. We can extend the usual operations from the real numbers to the complex numbers.

One topic, however, is never discussed: inequalities. Why not? Because it is impossible to extend the usual inequality relation (<) from real numbers to the complex numbers.

Here is a proof suitable for an advanced algebra class:

The inequality relation (<) between real numbers satisfies the three fundamental properties:

1. Any two distinct numbers can be compared; one of them is greater than the other (if $a \neq b$, then either $a < b$ or $b < a$).
2. The product of two positive numbers is positive (if $0 < a$ and $0 < b$, then $0 < ab$).
3. The product of two negative numbers is positive (if $a < 0$ and $b < 0$, then $0 < ab$).

Suppose that we try to extend the relation (<) to the complex numbers. Let's ask ourselves: Which is greater, i or 0? By the first property, either $0 < i$ or $i < 0$. If we assume that $0 < i$, then by the second property $0 < i^2$. If we assume that $i < 0$, then by the third property we again get $0 < i^2$. But $i^2 = -1$, a real number, and in real numbers we know that $-1 < 0$. Therefore, we have $i \neq 0$, but neither $0 < i$ nor $i < 0$ can be true, contradicting property 1.

Thus, it is impossible to extend the notion of inequality to complex numbers in a way that preserves properties 1, 2, and 3.

EASIER EXAMPLES, PLEASE

An experienced teacher may say, "Yes, these examples are interesting, and, yes, the proofs of impossibility work perfectly. But my students can't do this stuff about complex numbers or this Olympiad-style geometry problem."

Without going into a discussion about what students can do or cannot do (typical pessimism is, I believe, often misguided), let's just say that the issue of impossibility can be discussed within almost any topic.

Exploring a Straight Line

"Let's graph the equation $y = 2x + 1$." We can hardly find a more typical or more routine school assignment. Finding a few points on this line with integer coordinates is also not an unusual task. But now give the slightly different equation $y = 2x + 1/2$ and ask the same question about the points on the line with integer coordinates. Students will realize that finding such points is impossible and most likely will explain this impossibility: If x is an integer, then $2x$ is an integer, and $y = 2x + 1/2$ is not an integer.

This problem introduces the class of so-called Diophantine equations (named after Diophantus of Alexandria, an Ancient Greek mathematician), equations for which only integer solutions are acceptable. Some of these equations can require deeper observations than the problem above.

Here is just one example: Is it possible to find integers x and y such that $1620x + 1776y = 2014$? Or in terms of graphs: Does the line $1620x + 1776y = 2014$ contain points with integer coordinates?

The answer is no, because for any integers x and y, the number $1620x + 1776y$ is divisible by 3 (both 1620 and 1776 are multiples of 3), whereas 2014 is not divisible by 3.

A problem like this can be a perfect trigger for students' creativity. Ask them to make up similar problems for next week's test. They will enjoy the task.

Going Further

The same type of questions can be asked about any equation, not just about a Diophantine one. Here are some examples:

- Is it possible to find a solution to the equation $2(x + 1) = 3 + 2x$?
- Is it possible to compose a quadratic equation that has three solutions?
- Is it possible to find an equation with two unknowns, x and y, that has exactly one pair (x, y) as a solution?

The list can be easily extended. In some cases, an example shows that the answer is yes (although it may contradict some students' misconceptions, as in the last question). In other cases, however, the answer is no and requires a proof.

We can go even further. The same type of questions can be asked not only about equations but also about any mathematical situation. The typical set of school textbook assignments—"find," "solve," "evaluate," "construct"—should be enriched by the questions starting with "Is it possible . . . ?"

Going Beyond

The classical impossibilities in mathematics have always attracted professionals and amateurs by the deceptive simplicity of their formulations. It took more than three hundred years to prove the impossibility of positive integer solutions to the equation $x^n + y^n = z^n$ with $n > 2$, but research into the question led to great developments in mathematics.

The history of discovering impossibility in technology is no less dramatic than in mathematics. Centuries of unsuccessful attempts to create a perpetual motion machine provide one great example. Fiction and movies make people of all ages excited by the classic impossibility in physics—traveling faster than light or traveling backward in time.

Questions about the boundaries of possibility and about how those boundaries might change in different contexts can be very seductive. These questions deserve to be asked more often in high school mathematics courses. Questioning what is possible—or impossible—helps students view proofs as necessary and sharpens their logical-thinking skills.

ACKNOWLEDGMENTS

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JULIA VIRO, julia@math.sunysb.edu, is a director of the Math Learning Center at Stony Brook University in New York. Her primary interests are focused on teaching mathematics to in-service teachers.

Addressing the Standards for Mathematica

A two-part calculus activity uses true-false questions and a descriptive outline designed to promote active learning.

Mary E. Pilgrim

The Common Core State Standards (CCSS) provide teachers with the expectations and requirements that are meant to prepare K–12 students for college and the workforce (CCSSI 2010b). The Common Core State Standards for Mathematical Practice (SMPs) emphasize the development of skills and conceptual understanding for students to become proficient in mathematics and prepared for mathematics at the postsecondary level. These SMPs are the focus of this article:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning. (Common Core State Standards Initiative 2010a, pp. 6–8)

These practices reflect the desire for students to have a deep conceptual understanding of mathematics and the ability to connect ideas and approach problems in a thoughtful way. Students often see mathematics as a set of disjointed formulas and processes and attempt to work problems by mimicking solutions to other problems that fit the mold. Rather than encouraging students to follow a prescription, these standards push students to develop a bigger picture of mathematics. There is concern, as Russell (2012, p. 50) has noted, that teachers will be pushed to remain focused on high-stakes testing and treat the SMPs as a list of items that need to be checked off. This article proposes an alternative that parallels Russell's (2012) summary remarks: View the mathematical practices as interconnected skills and ways of thinking about mathematics rather than as separate ideas.

THE SMPs ALIGN WITH COURSE EXPECTATIONS

Whether at the secondary school level or postsecondary school level, goals for students enrolled in

Practice in a Calculus Class

calculus align with the Standards for Mathematical Practice. Students who successfully complete calculus should have a deep understanding of functions, the skills to analyze properties of functions using learned calculus techniques, and the ability to apply their learned knowledge to physical situations. Certainly most teachers would agree and do strive to meet such standards in the classroom. However, with dense content and fast-paced schedules, both secondary school and postsecondary school teachers, especially those new to teaching, can struggle with ensuring that all students have a deep understanding and the ability to apply their knowledge. This outcome becomes even more difficult with growing class sizes at colleges and universities.

This article will discuss the Standards for Mathematical Practice in the context of a particular activity designed for a first-semester calculus course at a state university in which the typical student is pursuing a career as an engineer or a scientist. The SMPs align well with the expectations of students in a first course in calculus at the postsecondary school level, raising the question: Why not design course materials that could be implemented in both high school and college calculus courses that reflect these practices and promote active learning?

As a start in this direction, a two-part activity that addressed the SMPs was used in a college Calculus I classroom. The class consisted of fifty-six students who were primarily college freshman and sophomores. The daily structure of the fifty-minute class included ten to twenty minutes of lecture, class discussion, peer-to-peer collaboration, and regular use of in-class activities. This article discusses the SMPs within the context of four questions posed to students and includes some student comments. For the first part of the activity, students worked individually for two to three minutes on a question, then spent two to three minutes discussing with a peer, followed by a class discussion before continuing to the next question. Class discussion time varied depending on the concepts and depth of discussion. The second part of the activity was done outside class.

PART 1: TRUE-OR-FALSE QUESTIONING RAISES COUNTEREXAMPLES

For the first two problems of the activity (see **problem 1** and **problem 2**), students were provided with either a theorem or a definition, followed by a list of statements. Students were directed to indicate whether each statement is true or false and explain how they know it is true or give a counterexample.

Students typically generated graphs to support their assertions. One student first drew a parabola

(see **fig. 1a**) to assert that statement 1(*a*) was true. However, after peer discussion, the students revised their response to say that the statement was false. To show that $f'(5) = 0$, students might first have drawn a quadratic opening up, with vertex at $x = 5$, to illustrate a minimum (see **fig. 1b**). Discussing possibilities with peers brought forward options that had not been considered previously, such as an increasing cubic without an extremum (see **fig. 1c**).

Common student responses to problem 1(*a*) noted the following:

- "There is a horizontal tangent line at $x = 5$."
- "False. It's reversed"
- "False. It doesn't tell us about the behavior around the point."
- "False. Need to know behavior around $x = 5$."

However, after peer discussion, students explained, "This graph [see **fig. 1c**] has a horizontal tangent but not a max/min" and "A function can have an instantaneous rate of change that is equal to zero at any point. It doesn't have to be a local max/min."

Problem 1: The First Derivative Theorem for Local Extrema

If f has a local maximum or minimum at an interior point c of its domain, D, and if f' is defined at c, then $f'(c) = 0$.

(*a*) True or false? If $f'(5) = 0$, then there is either a local maximum or local minimum at $x = 5$.
(*b*) True or false? If $f(-2)$ is a local maximum, then $f'(-2) = 0$.
(*c*) True or false? If $f(7)$ is a local minimum at an interior point of D and $f'(7)$ is defined, then $f'(7) = 0$.

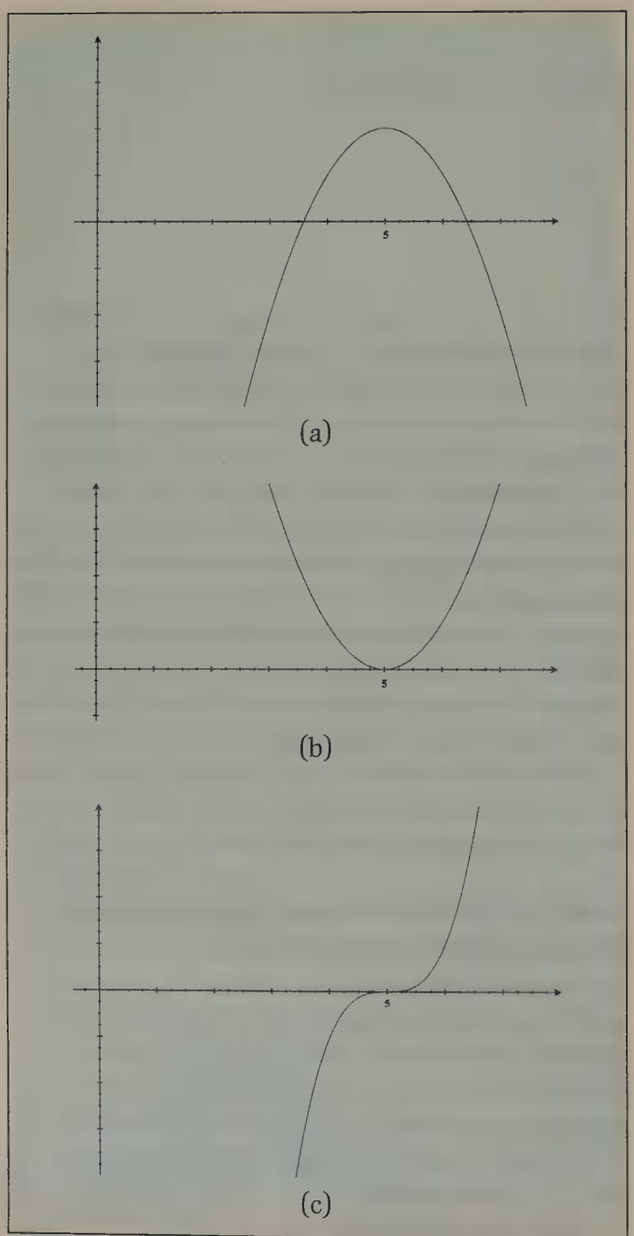

Fig. 1 Three graphs show a horizontal tangent at $x = 5$.

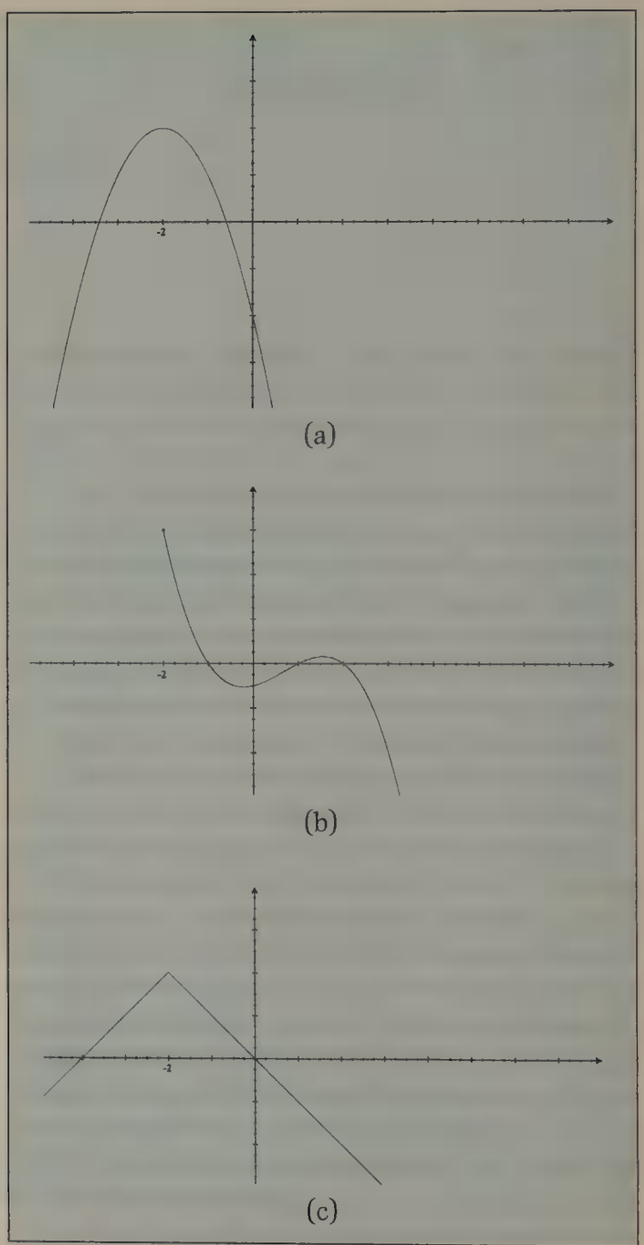

Fig. 2 Three graphs show a local maximum at $x = -2$.

Some students responding to problem 1(*b*) initially determined that the statement was true on the basis of graphs such as that shown in **figure 2a**. Additional graphs in **figure 2** prompted meaningful peer discussion:

- "False. Local max/min does not mean tangent line is zero at that point. Could be endpoints."
- "False. Doesn't say interior point. Could be endpoint."
- "False. At $x = -2$ could be nondifferentiable."
- "False. $f(-2)$ could be endpoint, or corner, or cusp."
- "Kind of true if the function is differentiable on its domain. If not, [then] not necessarily [true]."

Students determined that problem 1(*c*) was true by "pretty much using the theorem."

To answer problem 2(*a*), most students referred to their work in problem 1(*a*). Students responded that problem 2(*b*) was true, noting that "$f'(0)$ would be undefined due to the cusp" (see **fig. 3**).

Discussing Vocabulary and Revising Interpretations Allow Active Learning

These first two problems tie to SMPs 1, 2, 3, 5, 6, and 8. Students first need to read the problem and understand what is being asked. If students do not understand the question, then they will struggle with how to approach the problem. Not only are students asked to read and interpret a theorem, something with which most students struggle tremendously, but they also have to grapple with new vocabulary words. Do they understand *local maximum*, *local minimum*, *critical point*, and *cusp* and how such words relate to the derivative?

Understanding the problem can be even more challenging for international students and students whose first language is not English. The challenges of mastering a nonnative language are compounded by learning new and difficult mathematical vocabulary. However, learning new mathematical words and definitions is also difficult for native English speakers. Therefore, spending time discussing words, their meanings, and implications of theorems with examples and counterexamples is beneficial to all students.

Once students make sense of the problem, they can then reason through what is needed to solve the problem—the necessary concepts and their meanings. If the value of the derivative at a particular x-value is zero, does this imply that there is a local maximum or minimum at that point? What are the implications of the derivative equaling zero at a point? If there is a local maximum or minimum at a point, does this mean that the derivative is zero there? What does it mean to have a critical point? Constructing graphs and other examples of various situations that involve these concepts helps students better understand the problem, the relevant concepts, and what is needed to solve the problem.

Problem 2: Critical Points of a Function—A Definition

We have seen that local extrema could occur at endpoints, places where the tangent line is horizontal, and corners or cusps. An interior point of the domain of a function f where f' is zero or undefined is called a *critical point* of f.

(*a*) True or false? If $x = 1$ is a critical point of f, then $f(1)$ is a local maximum or local minimum.

(*b*) True or false? If there is a cusp at interior point $x = 0$, then $x = 0$ is a critical point of f.

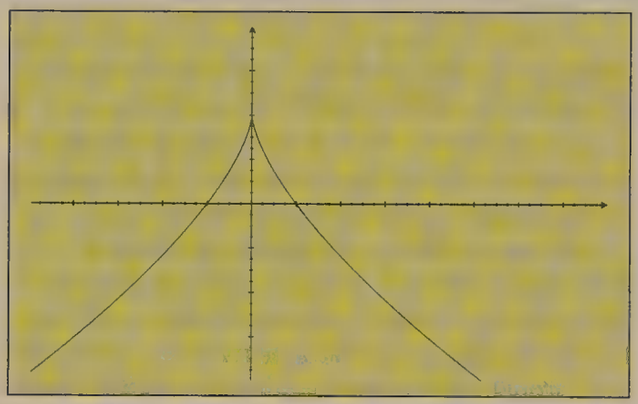

Fig. 3 Students relate the concepts of a cusp to the first derivative.

In addition, once students have written down their initial ideas and examples (or counterexamples), they are then able to discuss their ideas with others and construct arguments as to why a particular statement might be true or false. Peer-to-peer discourse gives students practice with using mathematical language and can lead to "aha" moments as they converse about and critique their different ideas and interpretations of the problem.

Creating initial solutions and ideas, communicating these thoughts with others, and then revising allow students to develop their understanding of concepts and develop more precise solutions. This process gives students an opportunity to be more active in their learning and interact more directly with the content—an opportunity not necessarily available from a lecture. In fact, as David Bressoud of the Mathematical Association of America bluntly states in his blog, "[S]itting still, listening to someone talk, and attempting to transcribe what they have said into a notebook is a very poor substitute for actively engaging with the material at hand, for doing mathematics" (2011).

Problem 3: Local Extrema

Draw the graph of a function that has a local maximum and local minimum not at an endpoint. What do you notice about the behavior of the slope of the tangent line around the extrema?

(*a*) For the local maximum: The slope of the tangent line on the left side is _____, and the slope of the tangent line on the right side is _____.

(*b*) For the local minimum: The slope of the tangent line on the left side is _____, and the slope of the tangent line on the right side is _____.

We say that f is *increasing* on an interval $[a, b]$, if $f' > 0$ at each point in (a, b). We say that f is *decreasing* on an interval $[a, b]$, if $f' < 0$ at each point in (a, b).

Problem 4: Absolute Extrema

For $f(x) = (x - 1)^2(x + 2)$ defined for all real numbers:

(*a*) Find the critical points.

(*b*) Find the intervals of increasing and decreasing.

(*c*) Identify all local extrema.

Do problem 4 without your calculator. Outline and describe the steps that enable you to solve this type of problem.

PART 2: A DESCRIPTIVE OUTLINE HELPS STUDENTS ARTICULATE STRATEGY

The second part of the activity is intended to get students to build on the knowledge they have and develop a method for testing for local extrema. These two problems (see **problem 3** and **problem 4**) tie to SMPs 1, 2, 3, 5, 6, and 8. Student comments are not included because students continued to work on this part of the activity outside class and then brought discussion questions to the following class period.

Students established general steps that they needed to work through problem 4. First, students needed the first derivative to identify the critical points where the first derivative is zero or undefined. Second, they tested values around the critical points to determine the intervals on which the function is increasing and decreasing. Third, they classified any extrema based on the behavior of the function around critical points. Finally, they found the extreme values by evaluating the function at the critical points where extrema occur. If the function had been defined on a closed interval, we could apply the extreme value theorem to identify absolute extrema.

Having worked through the previous set of true-false questions, students were able to make sense of the latter problems more quickly, but they still took their time to ensure that they understood the problem as well as what was needed to solve the problem. Students then began to construct arguments on the basis of what they knew about local extrema and discussed ideas with one another. This process allowed students to begin to develop a strategy that would lead them to a method for identifying local extrema.

As students worked through this part of the activity, they referenced their true-false responses frequently. Doing so allowed them to relate what they had just learned to the problem on which they were currently working. From their previously drawn graphs of local minima and maxima, students were able to determine that the derivative was positive before a local maximum and negative after a local maximum (and the opposite for a local minimum). By the end of class, they were starting to work with $f(x) = (x - 1)^2(x + 2)$.

The next class period began with a discussion of their ideas that lead to constructing a process for using the first derivative for testing for local extrema. Although SMP 4 is not directly addressed by this activity, students are now ready for applied optimization problems—an application of the knowledge just developed to real-world scenarios that would address modeling with mathematics (CCSSI 2010a).

EXAMINE DETAIL AND DEVELOP PROCESS: TYING THE TWO PARTS OF THE ACTIVITY TOGETHER

Students began each question of the activity individually and then worked in groups, after which a class discussion ensued. The activity was designed to get students thinking about definitions and the meaning of concepts. Too often students fall into the trap of assuming that extrema can occur only when $f'(x) = 0$ and forget about when $f'(x)$ is undefined or how the function is behaving at the endpoints (if there are endpoints). The true-false questions with counterexamples are meant to get students thinking more deeply about what is happening.

The finding local and absolute extrema aspect of the activity is intended to push students to apply the knowledge gained from the true-false questions and to think about the process of finding extrema. Students frequently want to be told what steps to follow; however, if students reflect on their knowledge about extrema and function behavior (increasing, decreasing, constant) and pick apart the definitions that they know, then they can then build the process on their own. The goal is for students to gain more understanding about extrema by discovering a process of finding them on their own rather than memorizing a prescribed set of steps that could potentially have less meaning for the student.

Activities such as this one are also a good way to get students to work in teams. Rather than going

straight to the teacher for the answer or getting stuck and giving up on a problem, students can discuss ideas with one another. Through this process of peer interaction, students can share their reasoning and critique one another's solutions, thus enabling students to gain a deeper understanding of the mathematical concepts. Multiple perspectives merge and begin to paint a more complete picture of the problem and solution.

One possible concern of teachers might be the amount of class time that such activities take. However, with this particular Calculus I class, because the activities and discussions helped students better understand the concepts, fewer in-class examples were needed.

DESIGN ACTIVITIES WITH THE SMPs

Calculus teachers routinely create worksheets compiled of problems, but creating an activity intended to push students to explore ideas and to think more deeply about concepts is not as simple. The goal of the Common Core's Standards for Mathematical Practice is to foster critical and creative thinking about mathematics. Therefore, a calculus activity should be designed with the SMPs in mind to allow students the opportunity to develop a better understanding of calculus concepts and the application of such concepts.

This calculus activity is an example that illustrates how the SMPs can be addressed simultaneously rather than as separate items on a bulleted list. If attention is given to designing curriculum that ties together the mathematical practices rather than treating them as a checklist, then students may develop deeper mathematical understanding and ways of thinking (Russell 2012).

In fact, although this activity was designed specifically for calculus, true-false questions are an example of a great way to get students of any level to explore mathematical concepts. True-false questions can provide an opportunity for students to work on vocabulary and the correct use of mathematical language. In addition, practice with speaking and writing mathematically in algebra and geometry courses can better prepare students for future work in a calculus course. Mathematics courses, regardless of level, all have shared goals—for students to develop a deeper understanding of mathematical concepts and ways of thinking mathematically.

This article is focused on the relevance of the SMPs to one specific topic in a calculus course, but there are numerous ways throughout mathematics courses (secondary and postsecondary alike) to implement the SMPs. Algebra and geometry are prerequisites for calculus, and the foundational goals align with those of calculus—pushing students to understand why and how and to not merely follow a recipe to a correct answer.

It is also important to note the growing diversity of mathematics classrooms at both secondary and postsecondary schools. For students whose native language is not English, a mathematics class can be quite difficult; mathematics is another language having vocabulary, grammar, and syntax. Activities that involve active learning and discussion give students a chance to practice speaking and writing mathematically while developing their understanding of the content.

In addition, any time teachers act as facilitators rather than lecturers, they get more information about their students. Discussions with students can provide an opportunity to identify gaps in knowledge and a venue for feedback. Results can inform teaching, and instruction becomes fluid and adapts to students' needs.

This article proposes that college mathematics instructors should attend to the SMPs as secondary school teachers do. The purpose is to help students develop a deeper understanding of mathematical concepts and push students to think more mathematically, which is what mathematics educators want from their students. A follow-up research study designed to examine the impact of implementing SMP-related tasks in postsecondary calculus courses is planned.

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MARY E. PILGRIM, pilgrim@math.colostate.edu, is an assistant professor in the mathematics department at Colorado State University in Fort Collins. Her current research interests include the education of preservice mathematics teachers and improving success rates in college calculus.

Lynn Mitzel and Mark Spanier

The Geometry of Tetris

When it was released in the mid-1980s, Tetris jump-started the video game craze, but many students of the current generation have never even seen this game, much less played it. Now, with the flood of mobile device applications, Tetris has made a comeback, and today's students have a chance to use it, too. We have found Tetris to be an engaging tool for high school geometry students to apply an isometry in context and to learn the composition of isometries. The game allows a player to rotate and translate moving pieces to create full rows anywhere on the screen.

We begin this activity by playing one of the many online versions of the game at http://www.netrover.com/~kingskid/tetris/tetris.html. This not only introduces the game but also generates student interest and game-winning strategies. Playing the game allows students to visualize and manipulate some of the isometries we have been learning in class; even though the students are playing a video game, they are learning content-specific applications of mathematics.

After students become familiar with the game, we present a mathematical description of a move, explaining the correct notation and sequence specific to compositions. The class discusses the required keystrokes to rotate or translate a piece; verbalization of the terms *rotate* and *translate* reinforces the students' use of academic language.

Superimposing a coordinate plane on the Tetris board connects the transformations to the coordinate plane (see **fig. 1**).

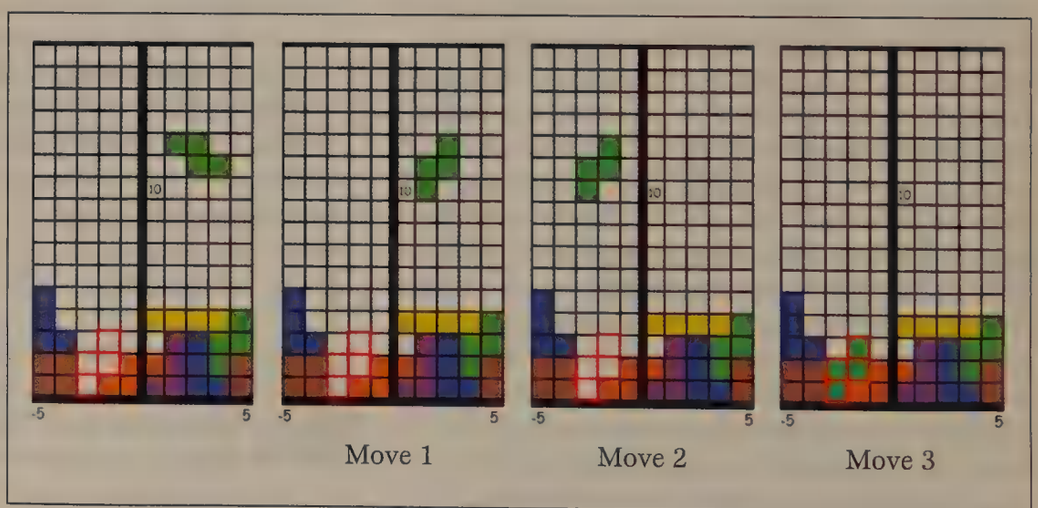

Fig. 1 Rotating 90° clockwise about the point (2, 11), then translating 4 units left, and then translating 9 units down would be written as $Translate_{\langle 0, -9\rangle} \circ Translate_{\langle -4, 0\rangle} \circ Rotate_{\langle 90° \text{ clockwise}, (2, 11)\rangle}$.

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#activities.

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Ruth Dover, *dover@imsa.edu, Illinois Mathematics and Science Academy, Aurora, IL; and* ***Patrick Harless***, *pdharless@gmail.com, University of Rochester (PhD student), Rochester, NY*

During this presentation, we emphasize that the mathematical notation allows a computer to process the moves. By shifting the learning responsibility from teacher to student in stages, we move students to independent practice using other Tetris examples superimposed on a coordinate plane. Students understand that many different correct compositions can move the piece to its proper final location.

The original game of Tetris allows a player only to rotate and translate pieces. Because we were also studying reflections, we asked students to determine how adding reflections to the game would change it. One student conjectured that only five game pieces instead of seven would be needed because some pieces are simply reflected images. Another student proposed that inclusion of reflections would allow a player to place pieces in spots that would be impossible using only rotations and translations. For example, the light blue piece in **figure 2** can be moved into the open location outlined in red using a translation to the right followed by a reflection across a horizontal line.

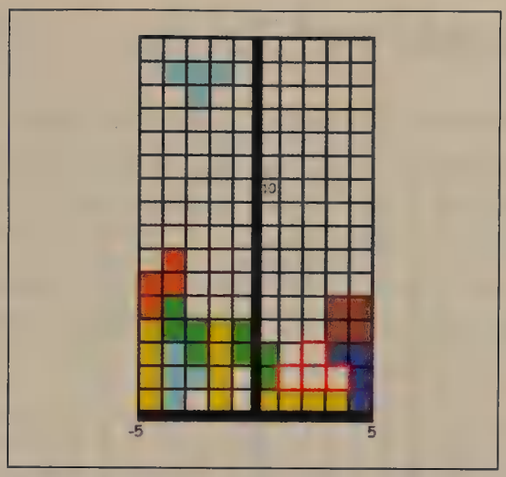

Fig. 2 The isometry $Reflect_{\langle y=8 \rangle} \circ Translate_{\langle 5, 0 \rangle}$ will place the light blue piece in a position that standard Tetris moves would not permit.

This activity could be extended by using pentominos (five connected squares connected perpendicularly) instead of the traditional Tetris pieces (four connected squares). With pentominos the game would have twelve playing pieces, making for a more challenging board.

The 1980s have come back in fashion, so Tetris might even be retro-cool for our students. More important, Tetris manages to engage students and spark their learning, making an abstract mathematical concept a real-world application.

LYNN MITZEL, lmitzel@hotmail.com, teaches mathematics at North High School in Fargo, North Dakota. She helps implement the Common Core State Standards at the middle school level and mentors new mathematics teachers.

MARK SPANIER, mark.spanier@ndsu.edu, is a mathematics graduate student at North Dakota State University in Fargo and a GraSUS Teaching Fellow in K-12 Education for Fargo and West Fargo Public School Districts. He is interested in using discovery learning to teach mathematics. The Graduate Student-University-School (GraSUS) is a program that pairs classroom teachers with university STEM students; the program is funded in part by NDSU and Fargo Public Schools.

Activities for Students

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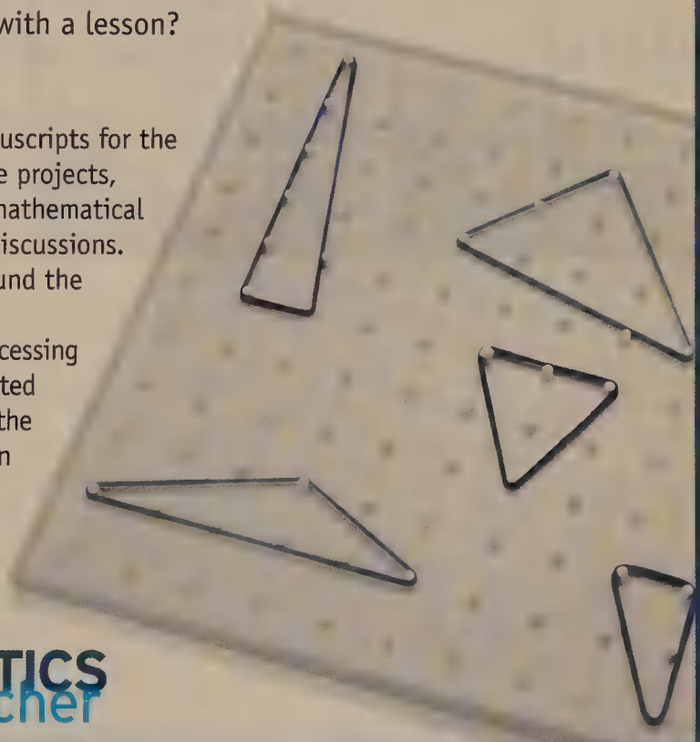

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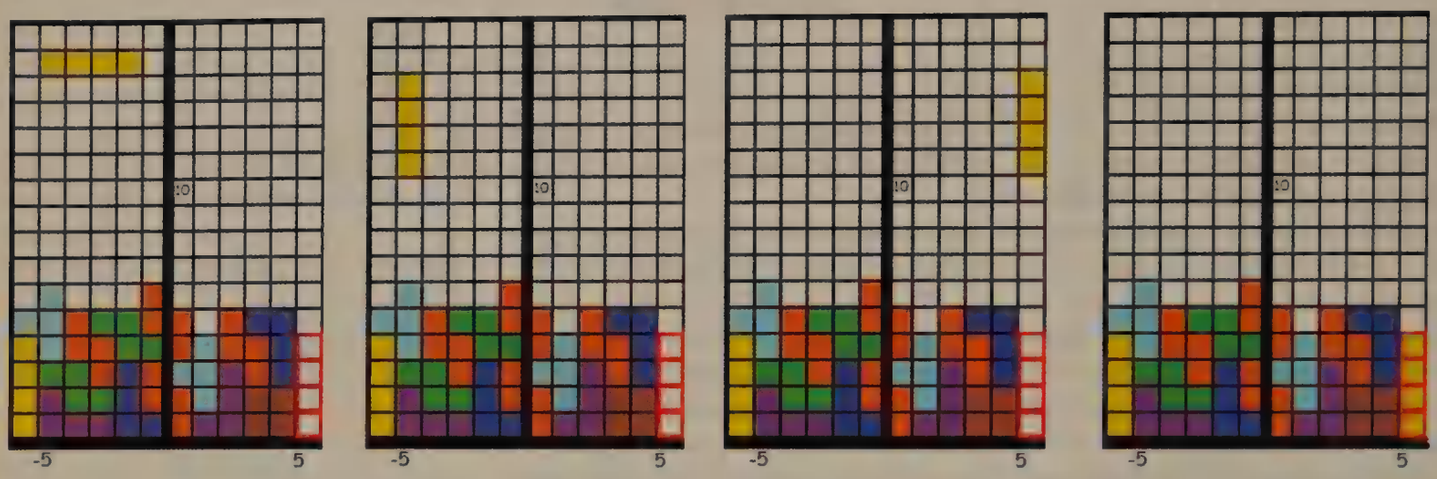
ACTIVITY 1: The Geometry of Tetris

Tetris is a game that has achieved worldwide popularity since its release in 1984. We will look at geometric transformations that can be used in this game. In the traditional Tetris game, a player is allowed only to rotate and translate objects. Use the link here to play a version of the original game: http://www.netrover.com/~kingskid/tetris/tetris.html.

Now that you have played the game of Tetris for a while, we will represent the rotations and translations of the game play using typical mathematical notation. Study the sequence of moves below required to place the falling yellow piece into the location outlined in red.



Move 1 Move 2 Move 3

Move 1: Rotate 90° clockwise about (–5, 14).
Move 2: Translate 10 units to the right.
Move 3: Translate 10 units down.

We would like to represent these moves as one statement (as a composition). We define an *isometry* as a transformation that preserves distance or length. We have learned that translations, rotations, and reflections are isometries. We also learned the following theorem:

The composition of two or more isometries is an isometry.

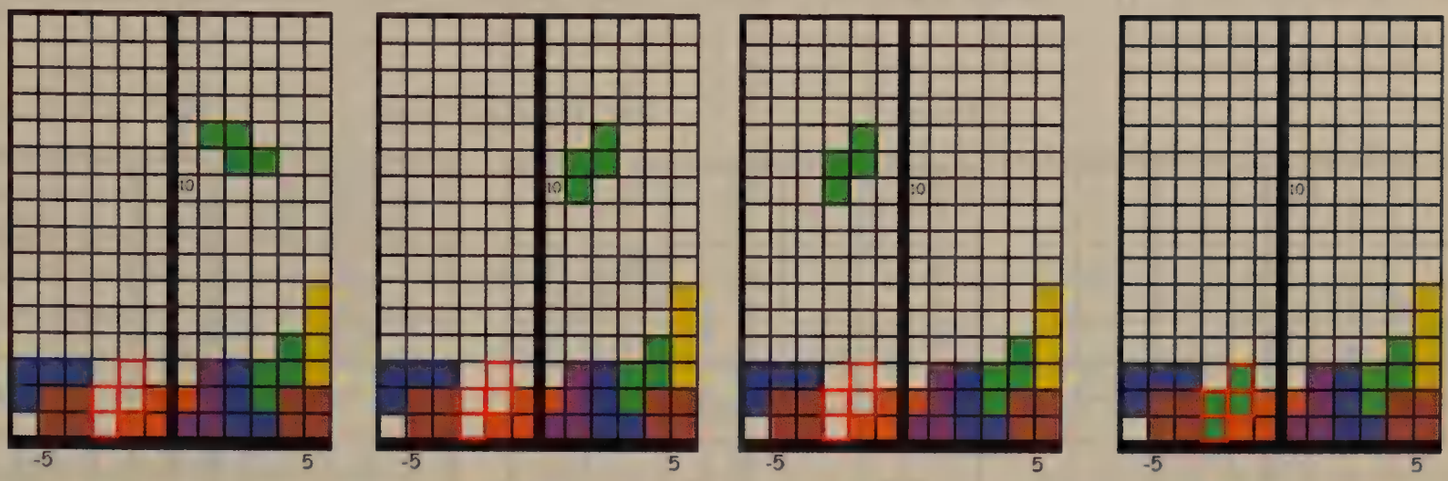
Using this theorem, we write the three moves shown above as a composition of isometries:

$$Translate_{\langle 0, -10\rangle} \circ Translate_{\langle 10, 0\rangle} \circ Rotate_{\langle 90^\circ \text{ clockwise}, (-5, 14)\rangle}$$

From the August 2014 issue of MATHEMATICS teacher

The Geometry of Tetris

For the Tetris moves shown below, write the sequence of moves as a composition of isometries.



Move 1 Move 2 Move 3

Move 1: _____
Move 2: _____
Move 3: _____
Composition: _____

The original game of Tetris does not allow for reflections. If we were to allow reflections as a possible move, we would need to identify the line of reflection. Suggest notation for the reflection isometry and label the moves it takes to place the light blue piece into its final position.

Move 1 Move 2 Move 3

Write the sequence of moves as a composition of isometries.

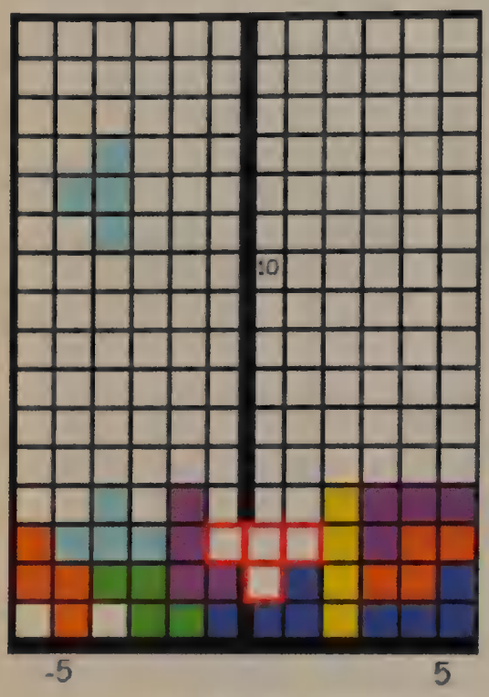
Move 1: _____
Move 2: _____
Move 3: _____
Composition: _____

From the August 2014 issue of MATHEMATICS teacher

The Geometry of Tetris

For each of the four Tetris boards numbered here, write the sequence of moves required to move the piece into the desired location. Try to use the minimum number of moves possible. Then write the moves as a composition of isometries.

Board 1



Moves: _____

_____

_____

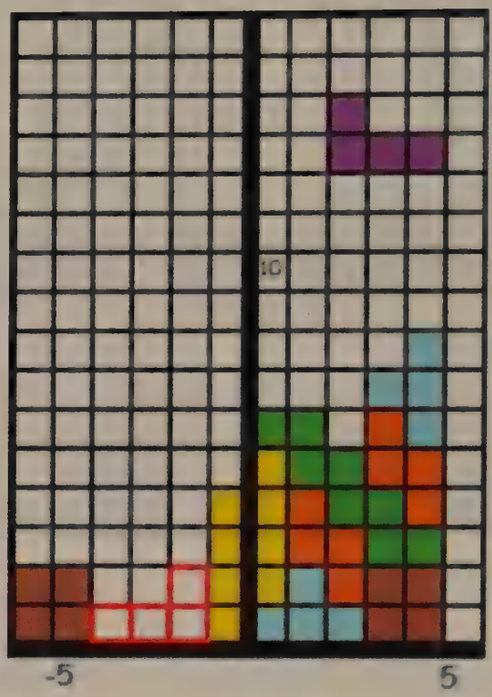
_____

_____

_____

Composition: _____

Board 2



Moves: _____

_____

_____

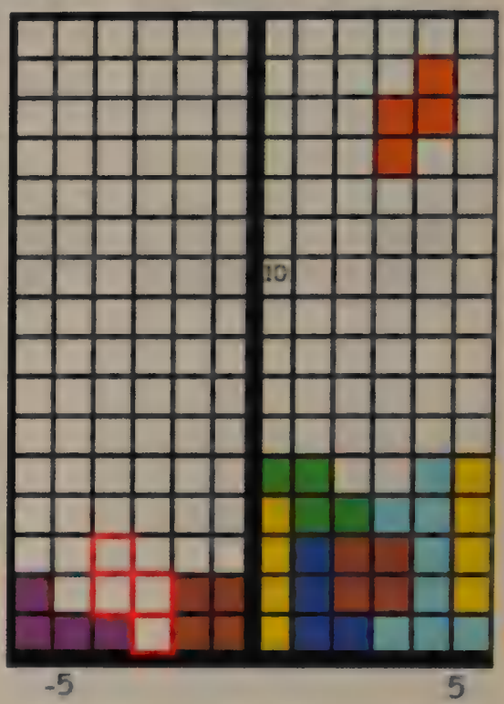
_____

_____

_____

Composition: _____

Board 3



Moves: _____

_____

_____

_____

_____

_____

Composition: _____

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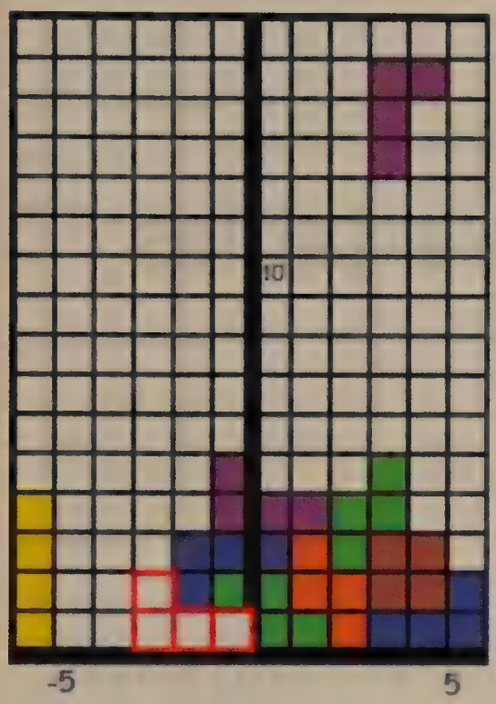
The Geometry of Tetris

Board 4



Moves: _____

_____

_____

_____

_____

_____

Composition: _____

Questions

1. In the online version of Tetris, which keyboard commands (buttons) allow you to translate a game piece?

2. In the online version of Tetris, which keyboard commands (buttons) allow you to rotate a game piece? What is the measure of that rotation?

3. The original game of Tetris does not allow for reflections. If we allow reflections, which of the Tetris pieces are unnecessary? Explain. (Hint: A picture may be helpful.)

4. For which of the Tetris pieces would a rotation not be useful? Why not?

5. What other moves would you like to be able to carry out in a game of Tetris? Explain.

6. Your friend says that she would like to add a glide reflection command to the game. How could she achieve a glide reflection using rotations, reflections, and translations?

7. Instead of using Tetris shapes (four connected squares), how would the game change with pentominos (five connected squares)? How many game pieces would be possible, and what would they look like?

From the August 2014 issue of MATHEMATICS teacher

Thomas G. Edwards and Kenneth R. Chelst

Linear Programming with More than Two Decision Variables

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

Department editors

James Kett, *j.gkett@gmail.com, Singapore American High School (retired), Singapore; and* ***Andy Benzing,*** *abenzing@wssd.org, Strath Haven High School, Wallingford, PA*

In a 1999 article in *Mathematics Teacher*, we demonstrated how graphing systems of linear inequalities could be motivated using real-world linear programming problems (Edwards and Chelst 1999). At that time, the graphs were drawn by hand, and the corner-point principle was applied to find the optimal solution. However, that approach limits the number of decision variables to two, and problems with only two decision variables are often transparent and inauthentic.

Additional decision variables give problems more authenticity but raise the question of obtaining an optimal solution. One approach—forming a matrix representation of the problem and applying the simplex method—is a tedious process for even modest-sized examples. Moreover, this method is beyond the reach of many high school students. However, technological tools such as Solver (Frontline Systems 2011; go to http://www.solver.com/), a standard add-in for spreadsheet applications, can be applied to solve the problem and also explore the sensitivity of the solution to changes in the parameters of the problem. MS Excel 2010 or 2011 software was used for this activity.

THE TWO-VARIABLE PROBLEM

We begin with a linear programming problem having two decision variables and two constraints. Some teachers with whom we have worked have begun this investigation by having students solve the problem algebraically with a careful graph before moving to the software-assisted solution. Following an algebraic formulation, we show how some analysis of the two-dimensional problem can be accomplished using Solver. Doing so establishes the basis of understanding necessary before proceeding to an extension of the problem having four decision variables and five constraints. After the larger problem has been formulated and solved, we discuss some "What if?" questions that the Solver software brings into play.

An example of the two-variable problem is the Snowboard Production problem:

> S4 Sno-boards is a sporting goods company that manufactures two styles of snowboards, Layout and Laser. The profit on each Layout is \$115, and the profit on each Laser is \$154.50. The manufacturing process for a snowboard has two main steps:

Let the decision variables be:

x_1 = weekly production rate of Layout snowboards
x_2 = weekly production rate of Laser snowboards
P = weekly profit

and the objective function to maximize:

$P = 115x_1 + 154.5x_2$

subject to constraints:

$44.5x_1 + 50x_2 \leq 11{,}000$ (preparation)
$41x_1 + 60x_2 \leq 12{,}000$ (finishing)
$x_1, x_2 \geq 0$ (nonnegativity)

(a)

	A	B	C	D	E	F
2	Product Mix; LP-Max					
3						
4		Layout	Laser			
5	Decision Variables	x_1	x_2			
6	Values					
7				Total Profit		
8	Objective Function Coefficients	115.00	154.50	0.00		
9						
10	CONSTRAINT COEFFICIENTS:			Amount Used		Amount Available
11	preparation time	44.5	50	0	≤	11000
12	finishing time	41	60	0	≤	12000
13						

S4-2x2

(b)

Fig. 1 Students will need to translate the algebraic formulation (**a**) into a spreadsheet formulation (**b**).

preparation and finishing. Each Layout requires 44.5 minutes of preparation and 41 minutes of finishing; each Laser requires 50 minutes of preparation and 60 minutes of finishing. The company has 11,000 minutes of preparation time and 12,000 minutes of finishing time available each week. What weekly production rates of each type of snowboard maximize profit?

An Algebraic Formulation

To formulate a problem like this, we define the *decision variables* (the variables over which there is some control), use the decision variables to define an *objective function*, and express any problem *constraints* in terms of the decision variables. In anticipation of the four-variable problem to follow, in this formulation we represent the decision variables as x_1 and x_2.

Obtaining a Solution with Solver

To obtain the optimal solution using Solver, we must translate the algebraic formulation into a spreadsheet formulation. Our experience has shown that students are more successful if they first generate an algebraic formulation (see **fig. 1a**). Later, we have noticed that many students writing the formulation directly in a spreadsheet format. **Figure 1b** shows a spreadsheet formulation of the Snowboard Production problem.

The Web enhancements accompanying this article (at **www.nctm.org/mt**) contain extensive directions for setting up and using Solver. At this point, we note that cells B5 and C5 contain merely labels

B6 fx

	A	B	C	D	E	F
2	Product Mix; LP-Max					
3						
4		Layout	Laser			
5	Decision Variables	x_1	x_2			
6	Values	96.77	133.87			
7				Total Profit		
8	Objective Function Coefficients	115.00	154.50	31812.10		
9						
10	CONSTRAINT COEFFICIENTS:			Amount Used		Amount Available
11	preparation time	44.5	50	11000	≤	11000
12	finishing time	41	60	12000	≤	12000
13						

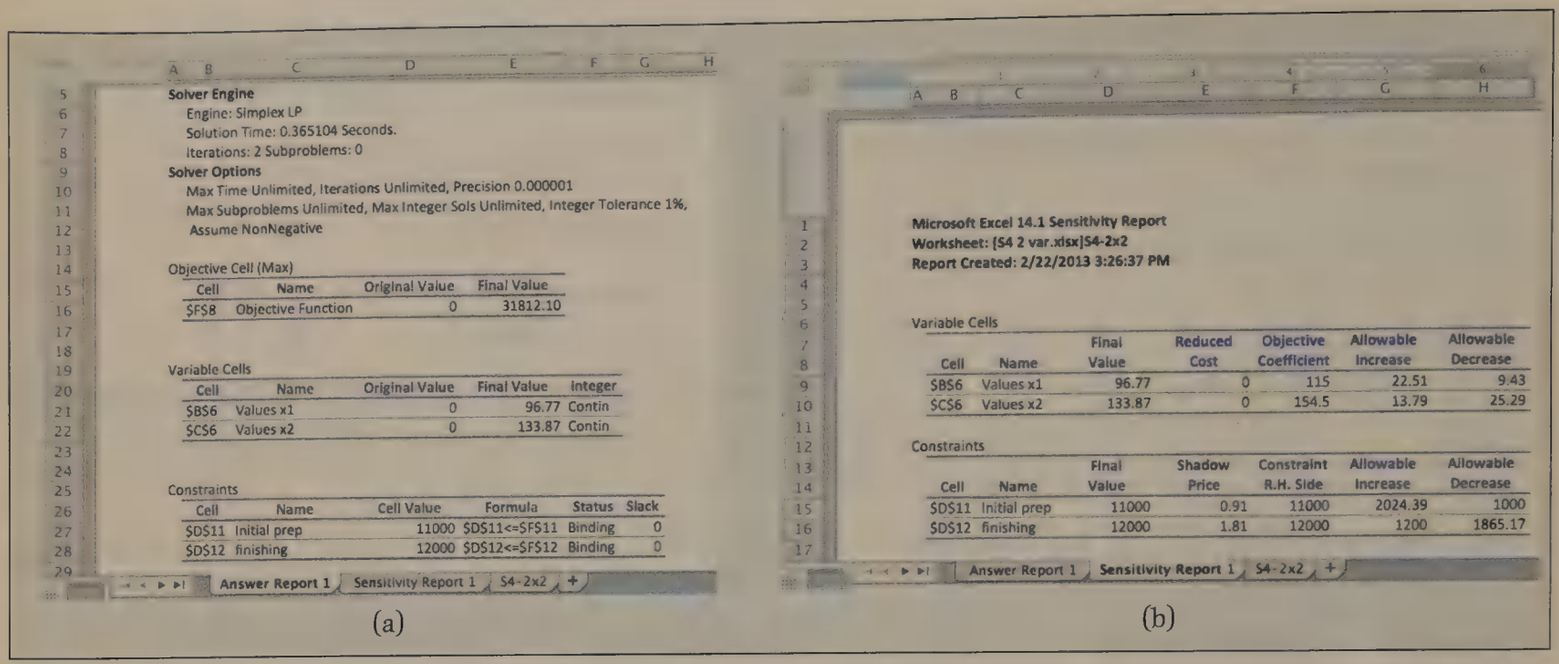
S4-2x2

Fig. 2 After Solver is executed, the spreadsheet looks like this.

for the decision variables. The actual values of the decision variables are carried in cells B6 and C6, and empty cells are interpreted as zeroes. Cells B8 and C8 contain the coefficients of the decision variables in the objective function, and cells B11, C11, B12, and C12 contain the coefficients of the decision variables in the two constraints. Moreover, cells D8, D11, and D12 do not contain numerical values. Rather, each contains a formula for computing the appropriate value using other cells in the spreadsheet. For example, cell D8 contains the value of the objective function for any particular set of values for the two decision variables—that is, D8 = B8*B6+C8*C6. The values in cells D11 and D12, the left sides of the two constraints, are similarly defined. The fact that these three cells contain formulas, not fixed values, can be a stumbling block for students.

The blank cells B6 and C6 are interpreted as values of zero. This explains why the values of the objective function (D8) and the left sides of the two constraints (D11 and D12) are reported as zero. Before running Solver, we like to have students explore this formulation by inserting particular values for the decision variables. Often, students will try a combination of values, such as $\{x_1 = 50, x_2 = 60\}$, that does not come close to expending all the resources in the constraints. When that happens, we like to challenge students to try to come up with increasingly more profit and enhance this activity. When students interact with the spreadsheet in this way, they get a feel for what the software can accomplish.

Finally, Solver can be applied to find the optimal solution. **Figure 2** shows the spreadsheet after Solver has been executed



Solver Engine
Engine: Simplex LP
Solution Time: 0.365104 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$8	Objective Function	0	31812.10

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Values x1	0	96.77	Contin
$C$6	Values x2	0	133.87	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$11	Initial prep	11000	$D$11<=$F$11	Binding	0
$D$12	finishing	12000	$D$12<=$F$12	Binding	0

Answer Report 1 | Sensitivity Report 1 | S4-2x2

(a)

Microsoft Excel 14.1 Sensitivity Report
Worksheet: [S4 2 var.xlsx]S4-2x2
Report Created: 2/22/2013 3:26:37 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Values x1	96.77	0	115	22.51	9.43
$C$6	Values x2	133.87	0	154.5	13.79	25.29

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$11	Initial prep	11000	0.91	11000	2024.39	1000
$D$12	finishing	12000	1.81	12000	1200	1865.17

Answer Report 1 | Sensitivity Report 1 | S4-2x2

(b)

Fig. 3 The Answer Report (**a**) and the Sensitivity Report (**b**) have been generated.

to find the optimal solution. Values for the weekly production rates of snowboards (the decision variables) and total profit (objective function) are presented in cells B6, C6, and D8. Notice also that for both constraints, the value of the left side is equal to the value of the right side. The right side tells how much of each resource is available, and the left side tells how much is used up by producing the optimal solution. When the two sides are equal, all the available resource is used up by the optimal solution.

Using the Answer and Sensitivity Reports

Figure 3 depicts the Answer and Sensitivity Reports generated by the Solver software. The Answer Report first presents the final value—that is, the maximum—of the objective function: \$31,812.10. Next comes the optimal solution: $x_1 = 96.77$ (Layout), and $x_2 = 133.87$ (Laser). Finally, there is information about the two constraints. Both are marked Binding, and for each constraint, a zero appears in a column headed Slack. But what does that zero mean? Remember that, in this problem, all the available resources are used up by the optimal solution. If that were not the case, any amount of the resource left over would be called Slack. The fact that there is no slack here explains the zero entries in the Slack column of the Answer Report.

The Sensitivity Report contains information about the decision variables and the constraints. For the variable cells, we see their final value, something called the reduced cost, the objective coefficient (in this case, the unit profit), and something called the allowable increase and allowable decrease. Allowable increases and decreases are also reported for each constraint as well as their final values (on the right side), and something called the shadow price. In this section, we will consider the allowable increase and decrease for the decision variable x_2, which refers to the Laser snowboard. The meanings of reduced cost and shadow price will be explored in our discussion of the larger four-variable problem.

Recall that the unit profit on Laser snowboards is \$154.5. Notice that in the Sensitivity Report under the Objective Coefficient column, this unit profit is listed for x_2. The next two columns to the right are labeled Allowable Increase and Allowable Decrease, and the values given are \$13.79 and \$25.29, respectively. For example, if the unit profit increased to \$164.50 or decreased to \$134.50, both those changes would be within the allowable range of \$129.21 (\$154.50 – \$25.29) to \$168.29 (\$154.50 + \$13.79). Encourage students to see what happens under either of these changes. For example, if the unit profit is changed to \$164.50 and Solver is rerun, students can observe that there is no change to the optimal production plan. What *does* change is the total amount of profit generated, which makes sense, because the profit on one of the snowboards produced has increased. Similarly, if the unit profit is decreased to \$134.50, the optimal solution is unchanged, but the total profit would decrease.

On the other hand, if the unit profit on Laser is changed to a value that lies outside the allowable range, the optimal production plan will change. Students can observe this fact by changing that unit profit to \$169.50. Then, S4 Snoboards would manufacture only the Laser snowboards.

A Word about Sensitivity Analysis

Teachers who have used material similar to this example have told us that at first the vocabulary was challenging for students and that students were often impressed by the sensitivity analysis. However, once students mastered the vocabulary and understood how changes in one parameter affected changes in the decision variables or the objective function, answering sensitivity analysis questions was straightforward. Having the real-world problem context to draw on made interpreting the values in the Sensitivity Reports much more meaningful for students.

Teachers also tell us that students saw the utility of the Sensitivity Reports in the ability to answer those analysis questions without rerunning Solver. They consistently tell us that their students were capable of and enjoyed doing sensitivity analysis of linear programming

problems using the Solver tool. Students were impressed with the wide variety of questions that could be answered using the information in the Sensitivity Report. They seemed to be particularly surprised that they could respond to "What if?" questions with very specific answers.

A FOUR-VARIABLE PROBLEM

We now expand the problem to four decision variables and five constraints. Previously, the only reason to restrict linear programming problems to two decision variables was to allow a graphical solution through the corner-point principle. In the real world, however, a typical model might have 100,000 constraints and nearly a million variables (Brown et al. 2001). That task is too sizeable to tackle in high school, but we believe that increasing the number of decision variables and constraints gives students a glimpse of what "real world" means.

Table 1 contains the unit profits per board for four snowboards that S4 Snoboards can produce, along with the time in minutes for each snowboard for five different production processes and the total time available for each process.

As in the two-variable problem, we believe students should begin by writing an algebraic formulation (see **fig. 4a**).

Figure 4b shows a spreadsheet formulation of the four-variable problem after it has been solved. Recall that the cells containing the values of the objective function and the left side of the constraints are all formulas, not numeric values. The optimal solution can be read directly from the spreadsheet—produce 84.51 Layouts, 70.42 Lasers, 84.51 Warriors, and no Feathers—generating a maximum weekly profit of $32,598.59. (If the spreadsheet cells are not formatted, they will display eight or more decimal places. To format the cells that will contain the computed numbers F8 and F11–15, choose **Format Cells-Number** and set the number of decimal places to **2**.) **Figure 5** displays the Answer and Sensitivity Reports for this problem.

Table 1 Unit Profit and Process Times for Four Types of Snowboards

	Type of Snowboard				
	Layout	**Laser**	**Warrior**	**Feather**	
Unit Profit	$115.00	$154.50	$142.00	$150.00	
Time in Minutes to Complete Tasks					**Total Minutes Available**
Shaping	24.5	28	22	23.5	6000
Cutting	10	8	7	7.6	2000
Silk-Screening	10	14	10	18	3000
Assembling	27	38	36	38	8000
Final Touches	14	22	15	17	4000

Interpreting the Answer Report

To drive home the concepts of binding constraints and slack, we have students verify the information in the Answer Report with the corresponding parts of the spreadsheet. For example, the Answer Report states that the shaping constraint is not binding and has a slack of 98.59. Looking at the spreadsheet, for the shaping constraint, we can see the inequality $5901.41 \leq 6000$. Because this resource is not completely consumed, the constraint is not binding. Moreover, subtracting the left side from the right side tells how much of this resource is left over and that amount is slack.

Let the decision variables be:

x_1 = weekly production rate of Layout snowboards
x_2 = weekly production rate of Laser snowboards
x_3 = weekly production rate of Warrior snowboards
x_4 = weekly production rate of Feather snowboards
P = the weekly profit

with an objective function to maximize:

$$P = 115x_1 + 154.5x_2 + 142x_3 + 150x_4$$

subject to the constraints:

$$24.5x_1 + 28x_2 + 22x_3 + 23.5x_4 \leq 6{,}000 \text{ (shaping)}$$
$$10x_1 + 8x_2 + 7x_3 + 7.6x_4 \leq 2{,}000 \text{ (cutting)}$$
$$10x_1 + 14x_2 + 10x_3 + 18x_4 \leq 3{,}000 \text{ (silk-screening)}$$
$$27x_1 + 38x_2 + 36x_3 + 38x_4 \leq 8{,}000 \text{ (assembling)}$$
$$14x_1 + 22x_2 + 15x_3 + 17x_4 \leq 4{,}000 \text{ (final touches)}$$
$$x_1, x_2 \geq 0 \text{ (nonnegativity)}$$

(a)

	A	B	C	D	E	F	G	H
2	Product Mix; LP-Max							
3								
4		Layout	Laser	Warrior	Feather			
5	Decision Variables	x_1	x_2	x_3	x_4			
6	Values	84.51	70.42	84.51	0.00			
7						Total Profit		
8	Objective Function Coefficients	115.00	154.50	142.00	150.00	$32,598.59		
9								
10	CONSTRAINT COEFFICIENTS:					Amount Used		Amount Available
11	shaping time	24.5	28	22	23.5	5901.41	≤	6000
12	cutting time	10	8	7	7.6	2000.00	≤	2000
13	silk-screen time	10	14	10	18	2676.06	≤	3000
14	assembly time	27	38	36	38	8000.00	≤	8000
15	time for final touches	14	22	15	17	4000.00	≤	4000
16								

Sensitivity Report 1 | Sheet1 | +

(b)

Fig. 4 The solution values of the four decision variables identified in the algebraic formulation (**a**) can be read directly from the spreadsheet (**b**).

Interpreting the Sensitivity Report

The allowable increase and decrease of the Feather snowboard, x_4, are interesting, because this snowboard would not be manufactured under the optimal solution. Notice that the allowable increase is \$0.90. We like to call students' attention to this fact and ask what the effect on the optimal solution would be if S4 Sno-boards could somehow increase the profitability of the Feather snowboard by more than \$0.90—say, by \$1 to \$151. Since the increase of \$1 is greater than the allowable increase, the optimal solution should change in some way. This change is easy to observe by changing the objective coefficient of x_4 in the spreadsheet to \$151, running Solver, and noting the resulting effect, if any. In this case, Feather enters the optimal solution, and the new production rates are Layout, 83.49; Laser, 61.83; Warrior, 43.85; and Feather, 47.83.

The allowable decrease for Feather production is even more interesting. To begin with, we believe that it is important to ask students what they think the value reported, 1E30, means. (Some students might correctly say that this is a computer representation of 1×10^{30}, but that misses the point.) The computer is telling us that the allowable decrease is really infinite! Now, why does *that* make sense? In a problem in which the goal is to maximize profit, decreasing the unit profit of an item that was not being produced would not change the optimal production plan.

Next, we consider the concept of shadow price. This is the amount by which the final value of the objective function would change if one additional unit of the constraint resource could be obtained. Students may test and verify this result. For example, the sensitivity report lists a shadow price of 1.44 for the cutting constraint. Now, if students increase the right side of the constraint by 1 to 2001 and run Solver again, they can observe the amount of profit increases by \$1.44.

Some of the constraints have zeroes reported as their shadow prices. This means that increasing the right side of the constraint equation will have no effect on the total amount of profit. Students may verify this as before, but the important thing is to ask them why this makes sense. Notice that neither of the constraints having zero shadow prices is binding. In other words, the associated resource is not being completely used up. So adding more of a resource that is not being fully consumed is going to add to the slack but not to the profit generated.

Finally, we investigate the reduced cost of a decision variable. We like to call attention to the fact that the reduced cost of three of the decision variables is zero, whereas the reduced cost of the remaining decision variable is –0.90. Then we ask students to make sense of this. Is there any characteristic separating one decision variable from the other

Microsoft Excel 14.1 Answer Report
Worksheet: [S4 4 var.xlsx]Sheet1
Report Created: 2/4/2013 1:51:21 PM
Result: Solver found a solution. All constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.717704 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Solve V

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$H$8	Objective Function	0	32598.59

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$6	Values x1	0	84.51	Contin
$C$6	Values x2	0	70.42	Contin
$D$6	Values x3	0	84.51	Contin
$E$6	Values x4	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$11	shaping	5901.41	$F$11<=$H$11	Not Binding	98.59
$F$12	cutting	2000	$F$12<=$H$12	Binding	0
$F$13	silk-screening	2676.06	$F$13<=$H$13	Not Binding	323.94
$F$14	assembling	8000	$F$14<=$H$14	Binding	0
$F$15	final touches	4000	$F$15<=$H$15	Binding	0

(a)

Microsoft Excel 14.1 Sensitivity Report
Worksheet: [S4 4 var.xlsx]Sheet1
Report Created: 2/4/2013 1:51:22 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Values x1	84.51	0	115	27.34	6.44
$C$6	Values x2	70.42	0	154.5	14.45	3.52
$D$6	Values x3	84.51	0	142	3.67	1.06
$E$6	Values x4	0	-0.90	150	0.90	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$11	shaping	5901.41	0	6000	1E+30	98.59
$F$12	cutting	2000	1.44	2000	75.21	378.38
$F$13	silk-screening	2676.06	0	3000	1E+30	323.94
$F$14	assembling	8000	3.41	8000	1126.44	777.78
$F$15	final touches	4000	0.61	4000	152.89	409.36

(b)

Fig. 5 The Answer Report (**a**) and the Sensitivity Report (**b**) are ready for students to interpret.

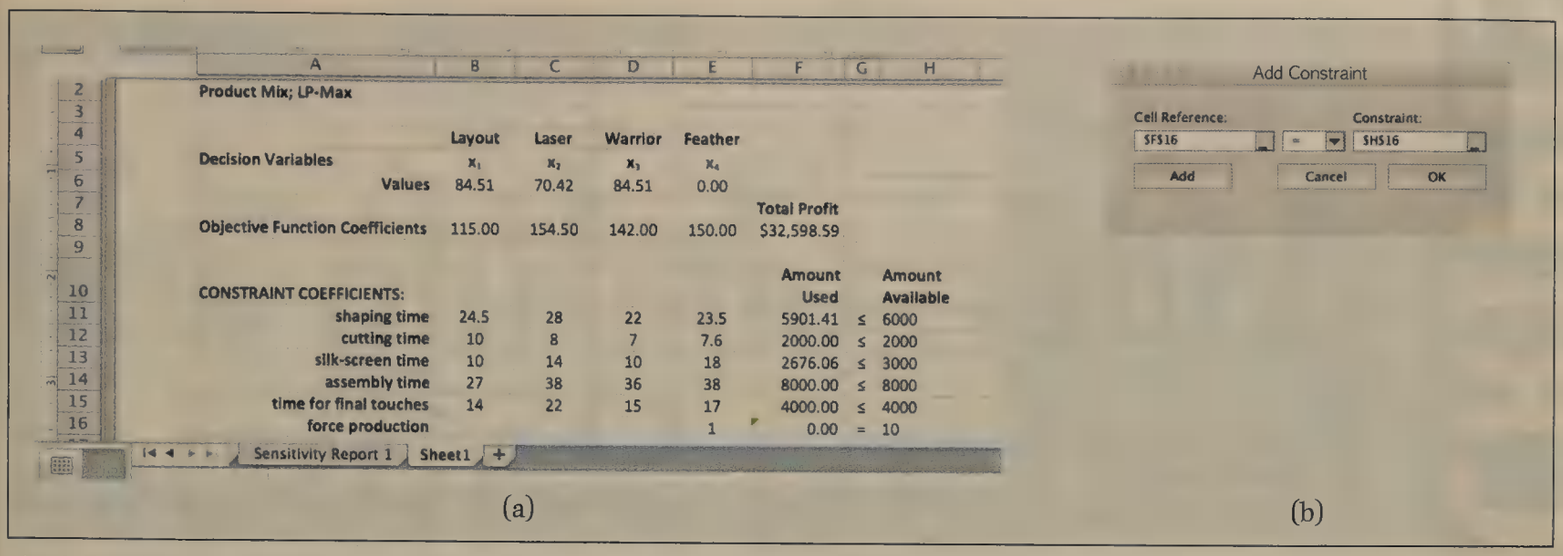

Product Mix; LP-Max
Layout Laser Warrior Feather
Decision Variables x1 x2 x3 x4
Values 84.51 70.42 84.51 0.00
Total Profit
Objective Function Coefficients 115.00 154.50 142.00 150.00 $32,598.59
CONSTRAINT COEFFICIENTS: Amount Used Amount Available
shaping time 24.5 28 22 23.5 5901.41 ≤ 6000
cutting time 10 8 7 7.6 2000.00 ≤ 2000
silk-screen time 10 14 10 18 2676.06 ≤ 3000
assembly time 27 38 36 38 8000.00 ≤ 8000
time for final touches 14 22 15 17 4000.00 ≤ 4000
force production 1 0.00 = 10
Sensitivity Report 1
Sheet1
(a)
Add Constraint
Cell Reference:
$F$16
Constraint:
$H$16
Add
Cancel
OK
(b)

Fig. 6 To gain an understanding of reduced cost, students will need to add the new constraint: $x_4 = 10$.

three? Notice that the decision variables with zero reduced cost are those that are being produced in the optimal solution, whereas the decision variable that is not being produced has the nonzero reduced cost. But why is the reduced cost negative? What might that mean?

At this point, we ask students to think of a scenario in which a company might force itself to produce units of an item that is not part of the optimal solution. Often, students will suggest that the company might do so for a good customer that regularly purchases that product. Pursuing this idea, we then ask students to write a constraint that would force S4 Sno-boards to produce 10 Feather snowboards, the product that is not in the optimal solution. Algebraically, the constraint is $x_4 = 10$. The spreadsheet formulation of this additional constraint is shown in **figure 6**; also shown is the dialogue box for adding the new constraint.

Students can again run Solver, noting that the profit is now slightly less than before. Why is this so? Forcing the production of 10 Feather snowboards has decreased the profit by about $9, which is precisely 10 times the reduced cost listed in the Sensitivity Report. Thus, a nonzero reduced cost of a decision variable is the amount by which the maximum profit will be reduced for each unit of that item that is produced. Most companies would not hesitate to take a special order of 10 units to satisfy a good customer when the result would be a reduction of only $9 on a total profit of over $32,000 for one week.

CONNECTING TO THE COMMON CORE STATE STANDARDS

The Common Core State Standards for Mathematics lists eight Standards for Mathematical Practice (SMPs). We believe that the classroom activities described here address each of them but with a clear focus on these: SMP 2—reasoning abstractly and quantitatively; SMP 4—modeling with mathematics; and SMP 5—using appropriate tools strategically (CCSSI 2010).

When students move back and forth between the problem context and its abstract representations, they are exhibiting the ability to contextualize and decontextualize, as suggested by SMP 2. When they apply mathematics to solve authentic, real-world problems, they are fulfilling the mathematical modeling vision of SMP 4. Finally, as they apply technology to better understand and solve problems, they bring to life the strategic use of appropriate tools mentioned in SMP 5 in a way that demonstrates that a computer is much more than a glorified computational device.

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THOMAS G. EDWARDS, t.g.edwards@wayne.edu, and **KENNETH R. CHELST,** kchelst@wayne.edu, are colleagues at Wayne State University in Detroit. They are interested in developing mathematics concepts within authentic real-world contexts as well as supporting the teaching and learning of mathematics in middle and high school using a variety of technological tools.

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MATHEMATICS Teacher

Debra K. Borkovitz

What's So Special about 3?

A few years ago, I encountered two different problems in which the number 3 played surprising roles. I found myself wondering, "Why 3? What's so special about 3?" Further investigation led to continuous extensions involving exponents, logarithms, a parametric equation, max-min problems, and some history of mathematics. As you read, pause to try the problems and play with the applets (the article's title is a big hint!)

THE PROBLEMS

The Biggest Product Problem

Start with a natural number n. Pick a set of natural numbers that add to n (repeats are allowed). Your score is the product of the numbers in your set. What is the highest possible score?

For example, if $n = 10$, some of your choices are $\{5, 2, 2, 1\}$ with product 20; $\{10\}$ with product 10; and $\{7, 3\}$ with product 21 (none of these is the biggest product).

This is a satisfying problem that I often use in a first mathematics course for elementary school teachers. It is overwhelming at first, but, after some effort, students find rules that together form a convincing solution.

The rules:

- Do not use any 1s.
- The set $\{2, 2\}$ is the same as $\{4\}$.
- Do not use any numbers bigger than 4. One way to justify this is to show that for $x > 4$, $2(x - 2) > x$, so $\{2, x - 2\}$ is better than $\{x\}$. Thus, there is a set of 2s and 3s that produces the maximal product.
- Now the final rule: With a set of 2s and 3s, use at most two 2s.The product of $\{2, 2, 2\}$ is 8, and the product of $\{3, 3\}$ is 9, so replacing three 2s with two 3s gives a bigger product.

Thus, in the biggest products, most factors are 3s, and the value of n mod 3 determines how many 2s are in the solution. If n is a multiple of 3, we do not use any 2s. If $n \equiv 2 \bmod 3$, then $n - 2$ is a multiple of 3, so we use one 2 and 3s for the rest. If $n \equiv 1 \bmod 3$, then $n - 1$ and therefore $n - 4$ are multiples of 3, so we use two 2s and 3s for the rest. For example, when $n = 30$, the biggest product is 3^{10}; when $n = 31$, it is $2^2 \cdot 3^9 = 4 \cdot 3^9$; and when $n = 32$, it is $2 \cdot 3^{10}$.

When I first saw this problem, my instinct was to use mostly 2s to maximize the exponent, but the 3s turned out to be key. What's so special about 3? Is anything deeper than the inequality $8 < 9$ going on?

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

Department editors

Maria Fung, *mfung@worcester.edu, Worcester State College, Worcester, MA; and* ***Ted Hodgson,*** *hodgsont1@nku.edu, Northern Kentucky University, Highland Heights, KY*

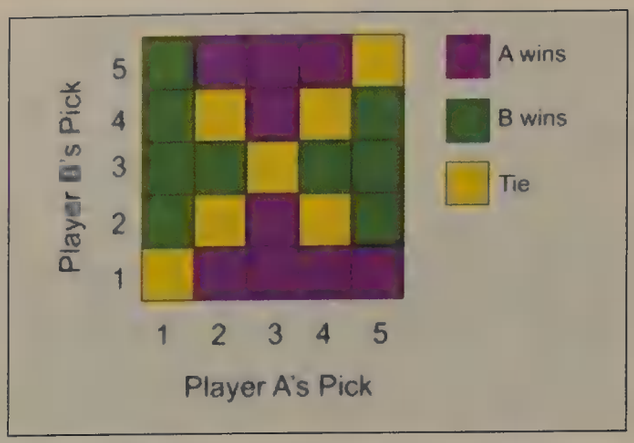

Fig. 1 The twenty-five possible outcomes of the finger game are colored to show the winner.

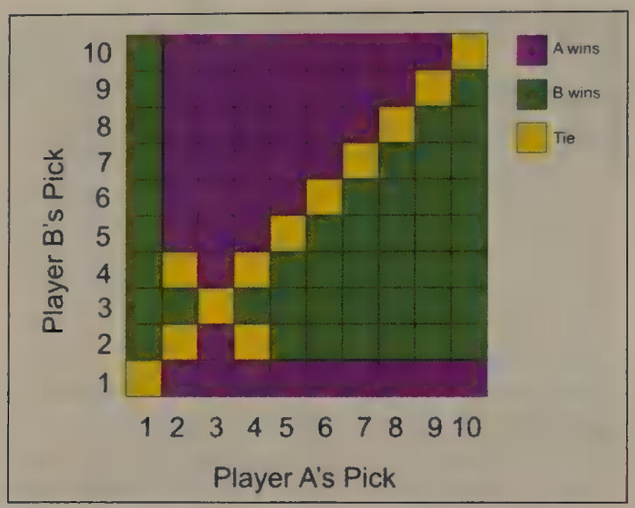

Fig. 2 The one hundred possible outcomes of the two-handed finger game are colored to show the winner.

Table 1 The Product $(n/k)^k$ of k Numbers That Sum to n

k	n/k	$(n/k)^k$
1	25.	25.
2	12.5	156.25
3	8.333333	578.7037
4	6.25	1525.879
5	5.	3125.
6	4.166667	5232.781
7	3.571429	7411.29
8	3.125	9094.947
9	2.777778	9846.4
10	2.5	9536.743
11	2.272727	8356.426
12	2.083333	6685.058
13	1.923077	4919.903
14	1.785714	3352.491
15	1.666667	2126.822

A Finger Game

Two players simultaneously hold out one to five fingers à la Rock, Paper, Scissors. The number of fingers on a player's hand is the base, and the number on the opponent's hand is the exponent. The player with the higher score wins. For example, if player A holds out five fingers and player B holds out two, then B wins because $2^5 > 5^2$. For a more advanced version, play with both hands. What's the best strategy?

This game, which I made up to help a ten-year-old make sense of exponential notation, replaced a much less interesting set of computational problems. **Figure 1** shows all possible results.

We discovered that if a player picks 3, she never loses (even if we play with both hands)! See **figure 2**.

What's so special about 3?

CONTINUOUS EXTENSIONS OF THE PROBLEMS

The Biggest Product Problem

A natural extension is to allow the factors to be any positive number. For example, for $n = 10$, we could choose $\{4.2, 2.8, 1.5, 1.5\}$, $\{10/3, 10/3, 10/3\}$, or $\{5 - \sqrt{2}, 5 + \sqrt{2}\}$.

With k factors, the product is maximized when all factors are equal. For example, with $n = 7$, in the original problem $\{2, 2, 3\}$ gave the maximal product of 12. In the extension with $k = 2$, $\{7/2, 7/2\}$ gives a product of 12.25; and with $k = 3$, $\{7/3, 7/3, 7/3\}$ gives a product of about 12.7.

When $k = 2$, proving that the product is maximized when the factors are equal is equivalent to a common max-min problem: What values of the length and width maximize the area of a rectangle with perimeter $2n$? For a noncalculus solution, let the sides be x and y, with $x \geq y$ and $x + y = n$, and let t be the distance from x to the mean of x and y. Thus, $x = n/2 + t$ and $y = n/2 - t$. Then $xy = n^2/4 - t^2$, which is maximized when $t = 0$ and $x = y = n/2$.

When $k > 2$, note that if two factors are unequal, we can create a bigger product by replacing them with two copies of their mean, and thus the maximal product has all factors equal. This argument is problematic because it assumes that a maximal product exists, but a more rigorous proof can be given using mathematical induction on k.

Thus, the maximal product with k factors that sum to n is $P = (n/k)^k$. Now our problem is to figure out which choice of k gives the biggest product. **Table 1** shows values for $n = 25$. The biggest product occurs when $k = 9$; each factor is $25/9 = 2.\overline{7}$. Note that the factors closest to 3 do not create the maximal product. When $k = 8$, we have $n/k = 3.125$, and $|3.125 - 3| < |2.\overline{7} - 3|$.

Table 2 The Ratio, n/k, When k Is Chosen to Maximize the Product

n	k	n/k
25	9	2.777777778
60	22	2.727272727
100	37	2.702702703
150	55	2.727272727
225	83	2.710843373
500	184	2.717391304
900	331	2.719033233

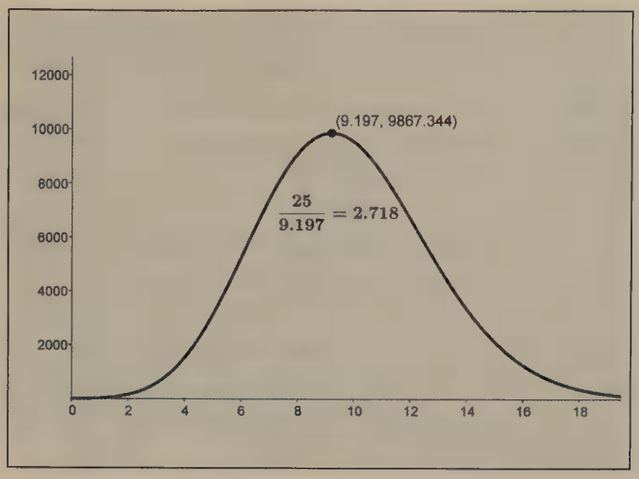

Fig. 3 The graph of $y = (25/x)^x$ appears to have a maximum when $(25/x) = e$.

Table 2 shows maximum product data gathered for many values of n. Note that the values of the factors, n/k, seem to be getting pretty close to one another and close to a famous number.

The spreadsheets are discrete and miss many values, so let's look at a graph of $y = (n/x)^x$, where n is fixed and x can be any positive real number. Allowing noninteger exponents helps us delve deeper but also takes away some connection to the Biggest Product problem.

Figure 3 shows the graph for $n = 25$. It appears that this function has a maximum value when $25/x \approx e$.

Let's look a little closer at our maximal product function $P(k) = (n/k)^k$. Taking logarithms and differentiating, we obtain $P'(k)/P(k) = \ln n - \ln k - 1$. Setting $P'(k) = 0$ and noting that $P(k)$ is never 0, we have $\ln(n/k) = 1$ or $(n/k) = e$ or $P = e^k$.

Thus, if it were possible, our best strategy would be to pick each factor equal to e. For $n = 7$, the biggest product would include $7/e$ "factors" of e, which would produce the product $e^{7/e} \approx e^{2.575} \approx 13.13$, which is bigger than any product we have found. Of course, a fractional number of factors does not make sense, so to solve the original problem we need to round.

What's so special about 3? It's close to e—that's what's so special!

Why are we better off choosing factors close to e? What's so special about e? How are well-known properties of e related to our results? To get more insight, we extend the finger game.

The Finger Game (Now the Paper Game)

Instead of using their fingers, players write down positive real numbers. Player A picks k, and player B picks x; player A wins if $k^x - x^k > 0$. The applet (go to http://www.geogebratube.org/material/show/id/4022) graphs the function $h(x) = k^x - x^k$; moving the slider gives a dynamic sense of how $h(x)$ changes when k changes (see **fig. 4**).

Note that $x = k$ is a root of $h(x)$. Another root usually appears on the graph of $h(x)$, unless k is close to e. When there are two roots, $h(x)$ is negative between them—that is, if one player writes k on his paper, he could win or lose, depending on his opponent's choice.

The graphs suggest something special about e that relates to our problems: e is the only positive value of k where k^x is always greater than or equal to x^k. If we write e on our paper, we never lose: e^x always beats or ties x^e, whereas for $k = 2$ we see that x^2 sometimes beats 2^x and so on. We get for free that $e^\pi > \pi^e$, something that is not immediately obvious. When $k = e$, $h(x) = e^x - x^e$, and basic calculus confirms that $h(e) = 0$ is in fact a minimum (although showing that it is the only minimum is harder).

How is this new property of e related to its more familiar properties? Additional insight comes from setting $f(x) = k^x$ and $g(x) = x^k$ and comparing their derivatives at $x = k$, where $h(x) = f(x) - g(x)$ has a root. We have $f'(k) = k^k \ln k$ and $g'(k) = k^k$. If $k < e$, then $\ln k < 1$, so $g(x)$ is growing faster than $f(x)$. There will be values of $x > k$ where $h(x)$ is negative and $g(x) = x^k$ is bigger than $f(x)$. On the other hand, if $k > e$, then $f(x)$ is growing faster than $g(x)$, and there will be values of $x < k$ where $g(x)$ is bigger. Thus, the only possible value of k where $g(x)$ is never bigger than $f(x)$ is $k = e$, where $f'(k) = g'(k)$.

Now we have a connection between the biggest product and one definition of e. What's so special about e? It's the only value of k where the derivative of k^x is k^x. Why is 3 key to the Biggest Product problem? Because 3 is the integer closest to the value of e and because the function $y = e^x$ is its own derivative. I find this result far from obvious.

A NEW GRAPH AND SOME HISTORY

For a different view of the paper game, we graph the implicit equation $x^y = y^x$ and shade accordingly to represent the related inequalities. The graph in **figure 5** is a continuous version of the graph in **figure 2**.

There are two branches where $x^y = y^x$. One,

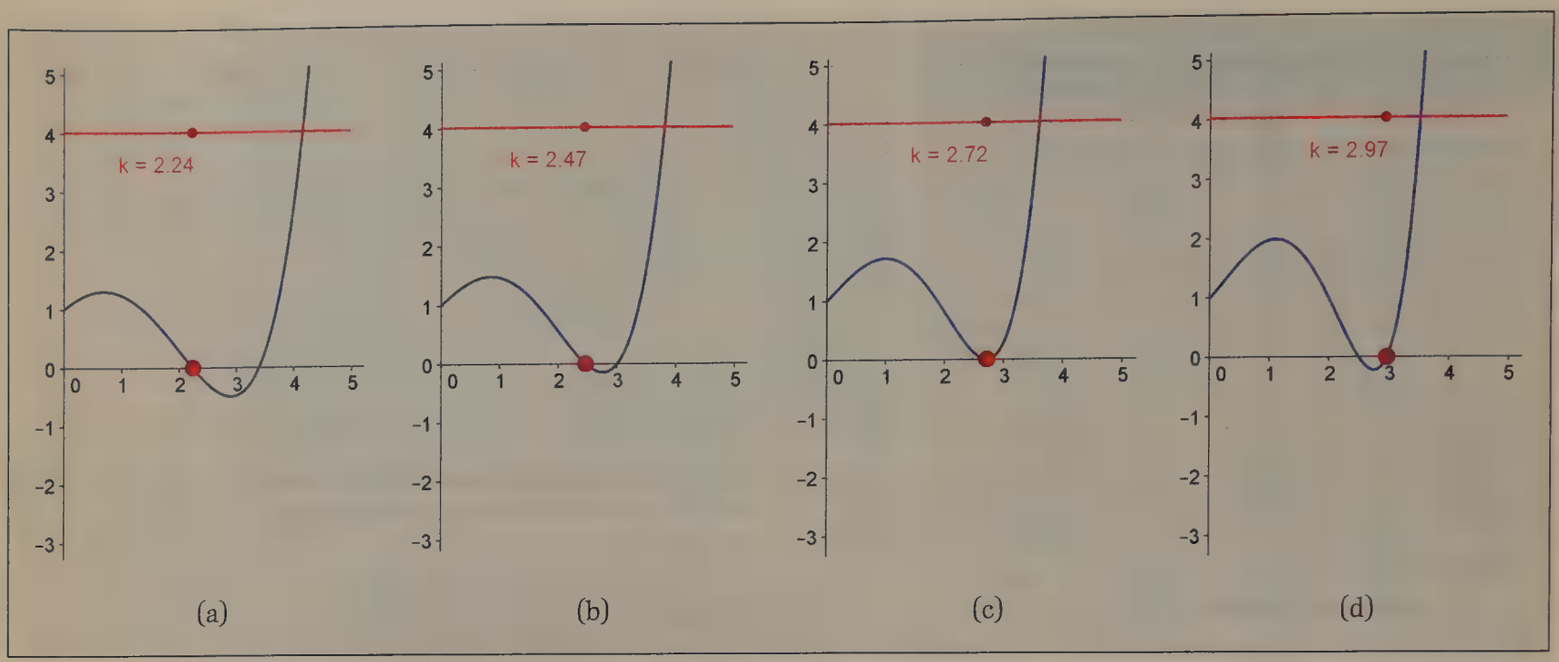

Fig. 4 Graphs of $h(x) = k^x - x^k$ show four different values of k. Use the related applet (http://www.geogebratube.org/material/show/id/4022) for a dynamic view of the graphs.

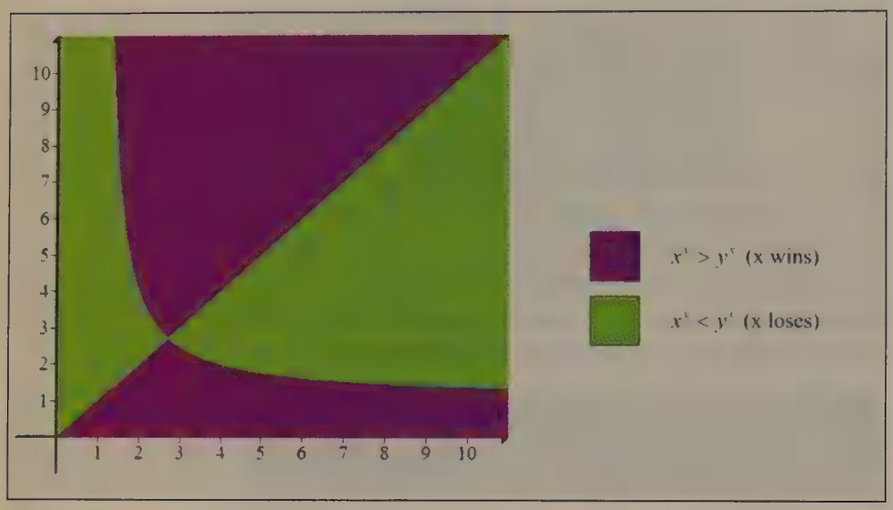

Fig. 5 This graph shows the two branches of $x^y = y^x$; it is a continuous version of **figure 2**.

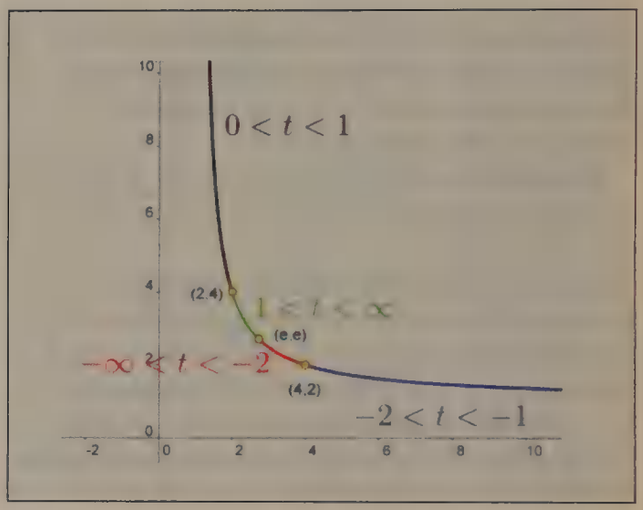

Fig. 6 The nontrivial branch of the graph in **figure 5** has the parametric equation $(x(t), y(t)) = [(1 + 1/t)^t, (1 + 1/t)^{t+1}]$, where $t > 0$ or $t < -1$. Use the related applet for a dynamic view of the graph.

$y = x$, is trivial; the other is more interesting. Can we find an equation for it? This is not a simple problem, but it is a relatively old one. Daniel Bernoulli and Christian Goldbach addressed it in correspondence in 1728, and Leonhard Euler also discussed it in some detail (Sved 1990). Their work links to another familiar property of e.

Goldbach solved $y^x = x^y$ using parametric equations and a clever trick: He set $y = ax$, where a is a positive real number, and then solved $x \log y = y \log x$ to obtain $x = a^{1/(a-1)}$, $y = a^{a/(a-1)}$. (Note that $a = 1$ gives the trivial branch of the graph.) Let $t = 1/(a - 1)$; then $x = (1 + 1/t)^t$, and $y = (1 + 1/t)^{t+1}$.

The nontrivial branch consists of all points

$$\left(x(t), y(t)\right) = \left(\left(1+1/t\right)^t, \left(1+1/t\right)^{t+1}\right),$$

where $t > 0$ or $t < -1$. Wow!

In this equation, x is the familiar expression for the balance on 1 unit compounded t times per year at 100% interest, and y is the same thing compounded once more. Another important and familiar result is that as $t \to \pm\infty$, $x \to e$ (as does y). Perhaps readers will see a way to connect the paper game to the properties of compound interest.

It is obvious that $x^y = y^x$ is symmetric about $y = x$; we can see the symmetry in the parametric representation by letting $T = -(t + 1)$; then $(x(T), y(T)) = (y(t), x(t))$. Move the sliders in the applet (go to http://www.geogebratube.org/material/show/id/3940) to dynamically see how different values of t create different parts of the graph in **figure 6**. (For a blog post related to these graphs, go to http://debraborkovitz.com/2012/01/kx-xk-slider-2/.) Note that $x = 1$ and $y = 1$ are asymptotes, an observation that we can prove using

L'Hôpital's rule and some manipulation. Also note, as suggested by the graph, that $2^4 = 4^2$ is the only nontrivial integer solution of $x^y = y^x$.

Returning to the paper game, we see that the intersection of the branches in **figure 5** is the point (e, e) and that $x = e$ "crosses" both branches of the curve at the same time, so a player who picks e never loses. For $k \neq e$, $k > 1$, the line $x = k$ intersects both branches, and there are values where each player wins. If both players pick values greater than e, then the player who picks the smaller number (as the base) wins (for example, we know that $8^9 > 9^8$). If both players pick positive numbers less than or equal to 1, the player who picks the larger number wins (for example, we know that $(1/2)^{1/3} > (1/3)^{1/2}$).

When a player picks a value between 1 and e, we need to use the nontrivial branch of the curve to determine the winner. The point (2, 3) is the only point with integer coordinates on the line $x = 2$ where x loses. This explains both my intuition to make as many values equal to 2 as possible in the Biggest Product problem as well as why its failure ($2^3 < 3^2$) is an anomaly.

BACK TO THE CLASSROOM

In the continuous version of the Biggest Product problem, we maximized the value of an exponential function while keeping the product of the base and exponent constant. In the continuous version of the Finger Game problem, we compared the values of $f(x) = k^x$ and $g(x) = x^k$. For both problems, we found that a base of e yielded optimal results; the proofs depended on the fact that the function $f(x) = e^x$ is its own derivative. In the discrete approximations to these continuous problems, the number 3 played a surprising role because of its proximity to e. (For another problem involving 3 and e, see Hayes 2008).

The discrete versions of the Biggest Product problem and the finger game are suitable for mathematics courses for elementary school teachers, but the continuous versions are generally too advanced. However, the process described in this article is important for future teachers to experience because so many believe that mathematics is only about finding right answers, preferably quickly and effortlessly.

I often ask students to articulate their intuition about a problem before working on the solution. Initially, students are reluctant to guess for fear of being wrong. With time, I hope, they will come to appreciate the pleasure that can come from allowing themselves to be wrong, to be surprised, to explore, and to learn something new.

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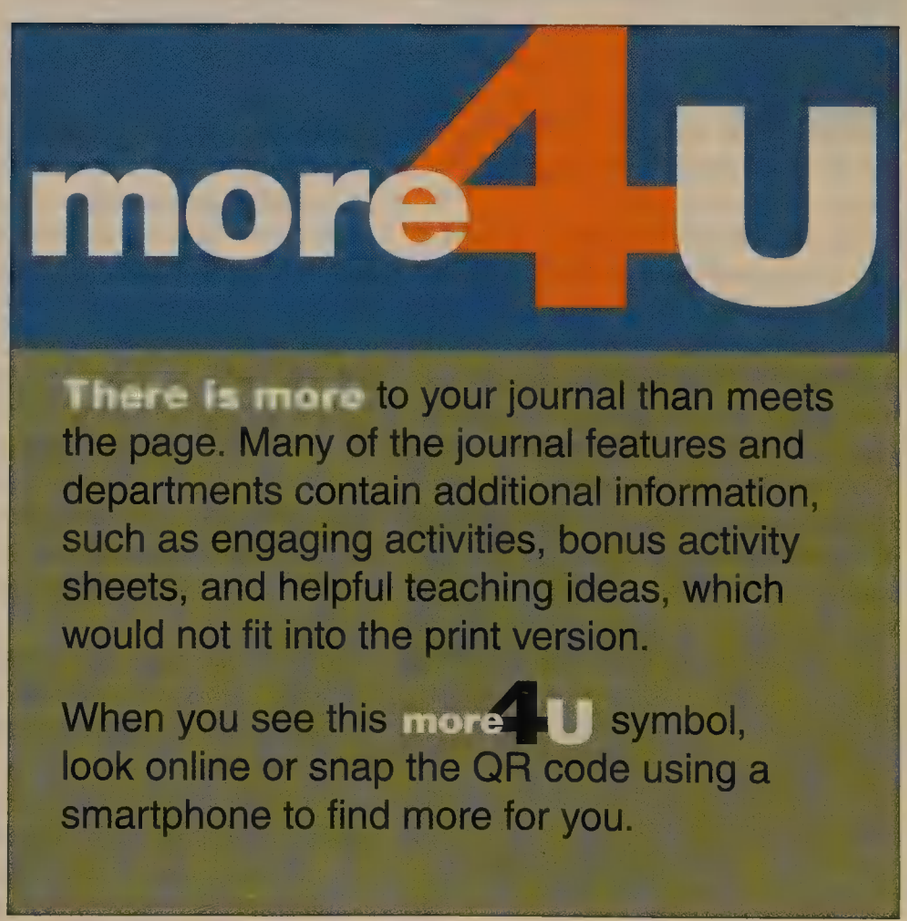
DEBRA K. BORKOVITZ, dborkovitz@wheelock.edu, is an associate professor of mathematics and co-chair of the Mathematics and Science Department at Wheelock College in Boston. She is particularly interested in developing curriculum for a mathematics major for preservice K–8 teachers.

Editor's note: For more analysis and classroom implementation of the Biggest Product Problem, see "A Rationale for Irrationals: An Unintended Exploration of e," by Nicholas H. Wasserman, *MT* March 2014, vol. 107, no. 7, p. 500–507.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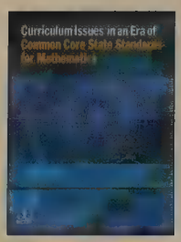
Curriculum Issues in an Era of Common Core State Standards for Mathematics, Christian R. Hirsch, Glenda T. Lappan, and Barbara J. Reys, eds., 2012. 219 pp., $37.95 paper. ISBN 978-0-87353-705-6. Stock no. 14319. National Council of Teachers of Mathematics; www.nctm.org.



The editors envision this book to be used as a part of a broader, district-level professional development, as a resource for focused building-level professional development, or in a master's degree course for teachers.

Section 1 traces the history of the Common Core State Standards for Mathematics (CCSSM). Section 2 outlines the curricular changes and emphasis found in elementary school, middle school, and high school. Two themes are reinforced throughout: (1) The Common Core State Standards are not a list of skills that students must acquire but a collection of "big ideas" that teachers need to help students understand; and (2) the best way to accomplish this is by teaching through problem solving.

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

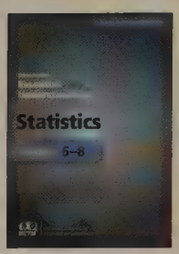
Section 3 contains three case studies—one each for elementary school, middle school, and high school—that focus on teachers developing and implementing their curriculum vision in light of the CCSSM. The same three authors write the three chapters, so there is considerable duplication. Nevertheless, the case studies are thoughtful.

Sections 4 through 7 were my favorite—full of lesson ideas that emphasize the Standards for Mathematical Practice, engaging discussions about the role of calculators and other technologies, practical suggestions and strategies for implementing statistical ideas, and ideas for ensuring that students receive a seamless mathematics education. These sections are also the most useful for classroom teachers.

Discussion questions are provided in each section. Thus, I believe that this book would be most beneficial to professional development facilitators or district curriculum leaders rather than individual classroom teachers. However, this book pushed me, as a classroom teacher, to reflect on how I understand and implement the Common Core State Standards for Mathematics in my classroom.

—Steve Phelps
Madeira High School
Cincinnati, OH

Developing Essential Understanding of Statistics in Grades 6–8, Gary D. Kader, Tim Jacobbe, Patricia S. Wilson, and Rose Mary Zbiek, eds., 2013. Foreword by Henry Kepner. 112 pp., $36.95 paper. ISBN 978-0-87353-672-1. Stock no. 13800. National Council of Teachers of Mathematics; www.nctm.org.



The book is divided into three sections; one covers content, the second discusses connections to other topics and vertical alignment, and the last examines questioning strategies and assessments. The audience is primarily in-service teachers of grades 6–8 mathematics. It could also be used with preservice teachers in a mathematics education course as well as with a Professional Learning Community of six to eight (or perhaps three to ten) mathematics teachers.

The book is organized extremely well by its "Big Ideas" and "Essential Understandings." The interspersed "Reflections" allow the reader to pause and explore the concepts being discussed. These would be ideal for use in a Professional Learning Community or a classroom of preservice teachers.

As a high school teacher, I wonder whether it would be possible to cite the Common Core Standards in some way, perhaps to have an index where the teachers could search by standard? Also, it might be beneficial to have a regular topic index that could be used to find certain topics such as *parameter* or *absolute mean deviation*. These are two suggestions that would be helpful for my own practice.

—Tess Rivero
Waynesville High School
Waynesville, OH

FROM OTHER PUBLISHERS

Activating Assessment for All Students: Differentiated Instruction and Informative Methods in Math and Science, Mary Hamm and Dennis Adams, 2012. 128 pp., $22.95 paper. ISBN 978-1-4758-0198-9. Rowman & Littlefield Education; www.rowman.com.

The first two chapters of this book are devoted to discussing differentiated

learning and informative assessment methods. Professionals looking for research-based information will be pleased, but high school teachers looking for information to use in their classes the next day will be disappointed. The authors clearly state that differentiated learning does not alter content but rather provides multiple pathways for students to access the material. Collaborative learning and multiple intelligences in relation to mathematics and science are also discussed along with many ways to use informative assessments.

The third chapter is devoted to discussing differentiated learning and informative assessment in the science classroom, and the fourth chapter covers the mathematics classroom. However, the activities provided in these chapters and in the final chapter of lesson plans are designed for elementary school and middle school students. Examples include classifying and comparing objects, place value, whole-number operations, fraction operations, and investigating pressure with a funnel and Ping-Pong® ball. Some of these activities could be adapted to a high school classroom if you are teaching developmental classes or need to incorporate those concepts into your curriculum. In addition, the ideas would be helpful if you are just starting out with differentiated learning and problem solving in your classroom. However, as noted, the ideas are not immediately implementable in a high school classroom.

—Gina Wilson
Early College Alliance
Eastern Michigan University
Ypsilanti, MI

The Best Writing on Mathematics 2012, Mircea Pitici, ed., 2012. 328 pp., $19.95 paper. Foreword by David Mumford, ISBN 978-0-691-15655-2. Princeton University Press; www.press.princeton.edu.

This is a collection of twenty-four essays on contemporary mathematics, its history and philosophy, and mathematics education. The essays are nontechnical, instead providing applications and general descriptions. The editor claims that the book is for lay people, but some background in college-level mathematics will allow deeper insights.

Diverse issues are addressed, including whether mathematics is invented or discovered; spheres in n dimensions; viewable spheres in photography; strange numbers; the distribution of prime numbers; the mathematics of dance; woven origami tessellations; mathematics pedagogy; mathematics in music; uses of variables; Augustus DeMorgan and his relationships with Sir William Rowan Hamilton and George Boole; Jean Bournoulli and the cycloid; issues in the philosophy of mathematics; and mathematical logic.

The book's strength is the way it gives insight into sophisticated issues at an accessible level. In most cases, important real-world applications of sophisticated mathematics are given. For example, in describing the birth of graph theory from Euler's Konigsberg bridge problem, Bruno, Genovese, and Improta go on to discuss the modern problem of routing a fleet of gasoline trucks from a central location to large number of gasoline stations. The extent to which authors are able to understandably articulate their topics varies.

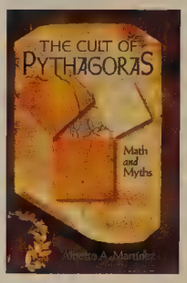
I recommend the book because of the vast number of applications of contemporary mathematics presented, the interesting historical accounts given, and the relevant discussions of issues in mathematics education.

—John E. Donovan II
Plymouth State University
Plymouth, NH

The Cult of Pythagoras: Math and Myths, Alberto A. Martinez, 2012. 288 pp., $27.95 cloth. ISBN 978-0-8229-4418-8. University of Pittsburgh Press; www.upress.pitt.edu.

One would hope that the mathematics community would know better than to blindly accept information without reason or evidence, yet in this book Alberto A. Martinez argues that this is exactly what we have done with the history of mathematics. Through his extensive research, Martinez explains that many historical accounts familiar to us are merely myths—void of any supporting documentation, grossly exaggerated and frequently embellished throughout the ages.

The Cult of Pythagoras: Math and Myths serves to set the record straight for two potential audiences. First, for those of us who consider ourselves

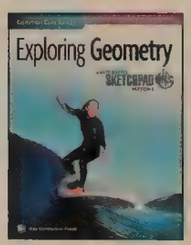


knowledgeable about the history of mathematics, this book aims to shake us out of our naivety by forcing us to reexamine how we know what we know and how we should distinguish fact from the fiction that so often substitutes for a proper history of mathematics. Second, for those who are not familiar with the history of mathematics, this book provides a compilation of historical examples that portrays mathematical knowledge as fluid, controversial, and ever evolving.

This book is probably best suited for secondary school and college-level mathematics instructors. Students would not likely be able to digest this book in its entirety, although teachers would surely find many examples to share that would make for fascinating classroom discussions.

—Brian Frasier
Merrimack College
North Andover, MA

Exploring Geometry with The Geometer's Sketchpad, Version 5. 2012. 349 pp., $39.95 paper. ISBN 978-1-60440-222-3. Key Curriculum; www.keycurriculum.com.



This publication shows the use of The Geometer's Sketchpad® to explore a wide range of geometric topics through construction and manipulation of visuals. The activities themselves do not mention the Common Core Standards, but at the back of book is a detailed list with the Common Core Geometry Standards and the activities related to each.

Each chapter begins with an overview of the activities and what the student

will accomplish within each activity. The activities presented provide step-by-step instructions for each exploration with questions and opportunities to explore the topics in more details. Sample answers are provided for the questions given throughout each activity and investigation as well as the "Explore More" questions, but they are not clearly labeled.

The questions in the activity are relevant, but the "Explore More" section challenges students by providing limited instructions, allowing students to engage beyond just describing what they see.

This book is useful for high school geometry teachers. The material is easy to follow, and the many activities enhance the text. Students will benefit most if they are able to work independently—that is, if all have access to computers and software.

—Nathalie Smalls
Georgia State University
Atlanta, GA

E-Z Algebra 2, Meg and Glenn Clemens, 2012. 600 pp., $16.99 paper. ISBN 978-1-4380-0039-8. Barron's Educational Series; www.barronseduc.com.

Barron's E-Z Algebra 2 is a resource book for second-year algebra that the authors describe as a "self-teaching manual." Despite this label, the authors make clear that the book is intended to serve as a companion or refresher resource rather than a textbook.

Each chapter begins with a set of learning objectives and a list of topics that are covered in the chapter. Each topic includes a brief description followed by several sample problems and solutions. Practice exercises are given at the end of each topic section and chapter, but neither answers nor solutions to these exercises are provided. Calculator tips are given at the end of the book.

In my teaching, I like to refer students to supplementary materials that they can use for home study, and I would refer students to this book as a supplemental resource because of the way in which the authors display the work required to solve a problem. The steps are neat and balanced, and I did not notice many skipped steps. The orderly and complete presentation of solutions will be easy for students to follow.

Although I appreciate the way in which solutions are written, the lack of answers or solutions for practice problems is a significant omission. If the book is intended for self-study, as stated, answers to practice exercises are important. Otherwise, how will students verify that their work is correct?

—Erika C. Bullock
Georgia State University
Atlanta, GA

Henri Poincaré: A Scientific Biography, Jeremy Gray, 2012. 616 pp., $35.00 cloth. ISBN 978-0-691-15271-4. Princeton University Press; www.press.princeton.edu.

This very dense, complete account of Poincaré's work in mathematics, physics, and philosophy as well as his career and his interactions with other mathematicians and scientists of the time is not for the mathematically faint of heart. Eight of the eleven chapters focus on specific aspects of Poincaré's work, which involved complex function theory, differential equations, topology, geometry, and a variety of topics in physics.

Along with the thorough treatment of Poincaré's mathematics is an equally thorough treatment of his philosophy of science and how he worked, thought, and interacted with other mathematicians and scientists. Teachers and students of mathematics would be interested in how Poincaré viewed intuition in mathematics. He struggled with mathematics and often made mistakes—a fact that would enlighten students who think that mathematics is easy or that every problem should yield a solution in twenty minutes or less. The book is rich with details of the debates between Poincaré and other philosophers and mathematicians, such as debates about non-Euclidean geometry and the role of axiomatic set theory.

The book is an endless source of interesting insights by Poincaré, including my favorite: "Thought is only a gleam in the midst of a long night. But it is this gleam that is everything" (p. 145).

I would recommend the book for mathematicians, mathematics educators, and philosophers in higher education who want a rich understanding of Poincaré, his work, and his times.

—Mary L. Garner
Kennesaw State University
Kennesaw, GA

Mathematics Galore! The First Five Years of the St. Mark's Institute of Mathematics, James Tanton, 2012. 271 pp., $50.00 cloth. ISBN 978-0-88385-776-2. Mathematical Association of America; www.maa.org.

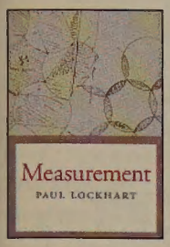
Mathematics Galore! is a collection of stimulating problems, extensions, puzzlers, and tidbits organized into twenty-six engaging essays around significant mathematical ideas (e.g., tessellations and partitions). The book is ideal for those who enjoy exploring mathematics concepts, problems, and connections. As Tanton notes, it is intended for high school teachers, student teachers, math club organizers, college faculty who prepare teachers or teach mathematics appreciation courses, mathematicians, mathematics majors, general mathematics enthusiasts, and mathematics circles.

The last essay is entitled "Weird Multiplication," and I explored the ideas presented here in a content course for preservice elementary school and middle school teachers. Although many of the methods (e.g., lattice multiplication and the area diagram) are ideas that I had discussed in previous courses, this book included new methods and extensions. I found that my students thoroughly enjoyed exploring the multiplication methods; in addition, by exploring these new algorithms, they were able to deepen their understanding of the underlying mathematics. Moreover, this book has inspired me to create several new projects for my courses for preservice teachers.

I was thoroughly engaged and stimulated by reading and working through the problems in *Mathematics Galore!* I highly recommend this book.

—Kelly McCormick
University of Southern Maine
Portland, ME

Measurement, Paul Lockhart, 2012. 416 pp., $29.95 cloth. ISBN 978-0-674-05755-5. Harvard University Press; www.hup.harvard.edu.



Mathematics education has become full of tricks and tools that avoid the true meanings of the concepts and can sometimes disorient students. Lockhart's *Measurement* is an answer to this growing trend and takes the reader on a journey to see how concepts should be viewed. There are no tricks, no shortcuts, no cheats—only great mathematics mixed with Lockhart's jovial writing. As a result, *Measurement* is a great resource.

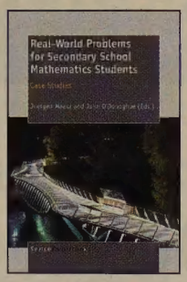
The concepts, although not easy, become more accessible with Lockhart's abandonment of complex mathematical terms and dialogue. The book is broken into two parts: (1) size and shape, an investigation of geometry; and (2) time and space, an investigation of trigonometry and calculus. Throughout both parts, Lockhart introduces an idea, explores it, and poses a question for readers to solve. The back of the book has no answers, no hints, nothing that would appease a student, just a blank page reminding us that we must work out the problems ourselves.

Lockhart's goal is to provide solid explanations for basic mathematics concepts and behaviors in a way that can be accessible by any student who puts in enough time and effort. He offers solid insight and ideas for secondary school teachers looking for better ways to explain ideas or provide explorative activities for students. Providing Lockhart's observations, problems, and encouraging words to students is a great way to give depth to lessons and student knowledge.

Measurement is sure to provide good insight and ideas that can push teachers and students to a deeper understanding.

—Christopher Cahill
The Shipley School
Bryn Mawr, PA

Real-World Problems for Secondary School Mathematics Students: Case Studies, Juergen Maasz and John O'Donoghue, eds., 2011. 292 pp., $49.00 paper. ISBN 978-94-6091-541-3. Sense Publishers; www.SensePublishers.com.



Intended for secondary school teachers and their students, this book presents fourteen in-depth descriptions and analyses of real-world situations that could be used in secondary school classrooms. Two additional chapters focus on the general notion of incorporating everyday situations into the mathematics classroom and using technology as a tool in the modeling process.

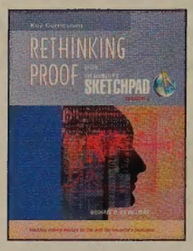
Topics range from digital imagery, space travel, and production processes to coding theory, polls and surveys, and the mathematics of eggs. A significant strength of most chapters is the extensive background, detail, and analysis provided for the topic. The nineteen-page chapter on coding theory, for instance, includes extensive mathematical discussion of bar codes, communication of digital images from space, and compact discs. In chapter 8, using NBA basketball to inspire modeling, the authors provide a multifaceted analysis of shot trajectory, devoting more than half the chapter to that.

Although many authors mention classroom uses for the topics and acknowledge the likely need for teachers to reflect carefully on how to shape their classrooms for these sorts of investigations, readers may be frustrated to find relatively little direct attention to classroom implementation. In the coding theory chapter, for instance, the "To Do in Class" section consists of only two paragraphs. We find a description of a project to consider, with no materials or examples provided—a not uncommon occurrence throughout the book.

If you are searching for a collection of real-world mathematics topics and are motivated to reinforce and extend your mathematics knowledge through careful study, this book is for you. The next step—classroom incorporation and implementation—will be your responsibility because these authors provide few specific suggestions or examples.

—Roger Day
Illinois State University
Normal, IL

Rethinking Proof with The Geometer's Sketchpad Version 5, Michael D. Devilliers, 2012. 214 pp., $29.95 paper. ISBN 978-1-60440-279-7. Key Curriculum; www.keycurriculum.com.



This collection of thirty-one geometry lessons using The Geometer's Sketchpad® focuses on developing the concepts of proof in mathematics through geometry activities. These lessons can be used in the high school geometry classroom as well as in a college-level mathematics course. Each lesson provides easy-to-follow directions and explanations along with corresponding files to be downloaded from the GSP website. These files can be used with GSP version 5 and the iPad app Sketchpad® Explorer.

Rethinking Proof is an excellent resource to assist any high school or college teacher when teaching proof in geometry with such concepts as properties of quadrilaterals, colinearity, and perpendicular bisectors, to name a few. The author also provides opportunities for further explorations to most lessons and an answer key to each activity. However, an average high school geometry class might find it difficult to complete the entirety of a lesson because of time constraints. Therefore, teachers might need to adapt the lessons to meet their individual classroom needs.

Many high schools are integrating technology into their instruction—specifically, technology that allows students to model the mathematics through real-life situations and broaden their depth of understanding. This resource can be an effective collection for geometry instruction. The concepts of proofs are well organized, and the activities are designed in a clear, well-structured manner. High school mathematics teachers and college mathematics instructors who teach a course on proof or geometry will find this book helpful.

—Raeann Kyriakou
Xaverian High School
Brooklyn, NY

MY FAVORITE lesson

Cindy Kroon

Arctic Search and Destroy

"You sunk my battleship!" Many of us have fond memories of the Milton Bradley game Battleship®. Using rectangular coordinates to identify and sink an opponent's ships remains a classic childhood pastime. A similar activity can be used to help students as they learn to plot positions in the polar coordinate system.

The Common Core State Standards for Mathematics (CCSSM) suggest that students "[r]epresent complex numbers on the complex plane in rectangular and polar form, and explain why the rectangular and polar forms of a given complex number represent the same number" (CCSSM Standard N-CN.4).

In the rectangular coordinate system, the location of a point is uniquely designated by an ordered pair (x, y). For instance, $(1, \sqrt{3})$ represents the point whose x-coordinate is 1 and whose y-coordinate is $\sqrt{3}$. There is no other way to use rectangular coordinates to designate this point.

In the polar coordinate system, the location of a point is designated by an ordered pair (r, θ), with radius length r and angle measure θ, but an infinite number of ordered pairs can represent the same location. The rectangular coordinate point $(1, \sqrt{3})$ can be converted to polar form as $(2, 60°)$, $(-2, 240°)$, $(2, -300°)$, $(-2, -120°)$, and infinitely many other ordered pairs of the forms $(2, 60 + 360n°)$ and $(-2, 240 + 360n°)$, where n is an integer. Expressing the angle in radians introduces an equivalent list of possibilities—an element of challenge, or possibly confusion, for students.

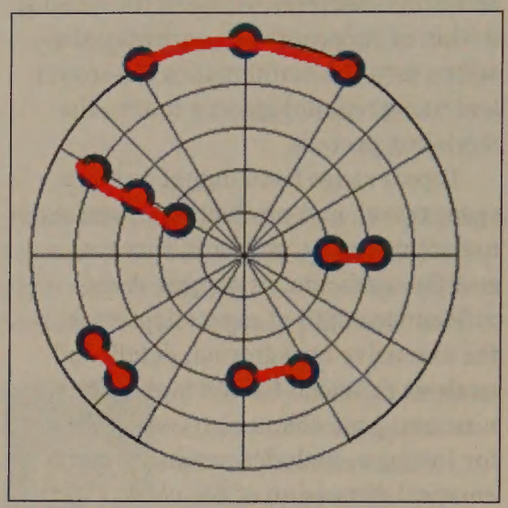


Fig. 1 Ships are represented by two or three connected points.

Graphing on the polar plane is typically introduced in a trigonometry or precalculus course and provides an essential building block for several advanced concepts. Cylindrical and spherical coordinates build on the basic concepts of the polar coordinate system by adding a third dimension. Calculation with vectors requires many of the concepts developed in polar graphing and promotes connections to physics applications.

Students frequently have difficulty making the transition from the rectangular coordinate plane to the polar plane. An engaging practice mechanism is essential to student mastery. An activity sheet, Arctic Search and Destroy (available at **www.nctm.org/mt**) was inspired by the classic Battleship game and adapted for use in the polar plane.

Students use polar coordinates (r, θ) to secretly place five ships on a polar plane (see **fig. 1**). Ships can be placed on radial segments or arcs of the polar plane. Teachers can designate allowed angle measures as either degrees or radians to vary the difficulty level of the game. A similar version of the game was published many years ago in *Mathematics Teacher* (Camp 1995).

This activity could be differentiated in many ways. For instance, students could use angle restrictions, such as $-2\pi < \theta < 2\pi$, that allow for multiple representations of a single location. As an extension, students could be required to name both polar and rectangular coordinates as they try to locate their opponent's ships. Alternatively, guesses could be made in rectangular form and recorded in polar form.

Playing Arctic Search and Destroy provides an engaging way for students to practice graphing polar coordinates. Students who might balk at completing a worksheet participate enthusiastically and quickly become proficient in polar graphing.

The game's title was suggested by a student: Arctic (Polar) Search and Destroy. (Get it?)

REFERENCE

Camp, Dane R. 1995. "Sharing Teaching Ideas: Starship." *Mathematics Teacher* 88 (2): 113–15.

CINDY KROON, Cindy.Kroon@k12.sd.us, teaches mathematics at Montrose High School in Montrose, South Dakota. She is interested in problem solving and connecting mathematics and science. She gives professional development workshops and is president-elect of SDCTM.

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For instructions and a customizable activity sheet, go to **www.nctm.org/mt**.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Department editor
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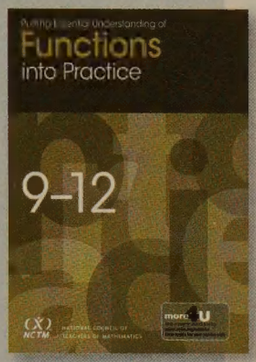

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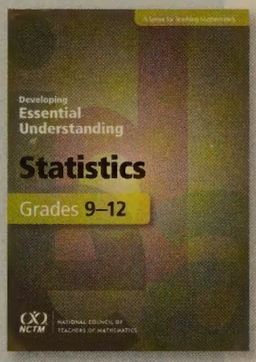

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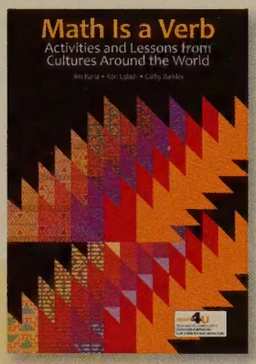

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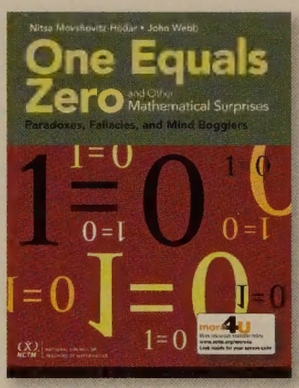

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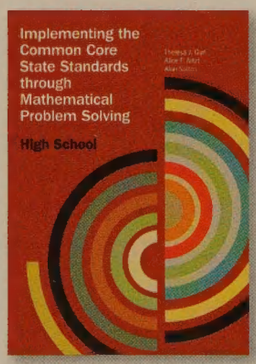

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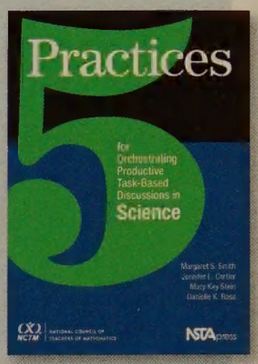

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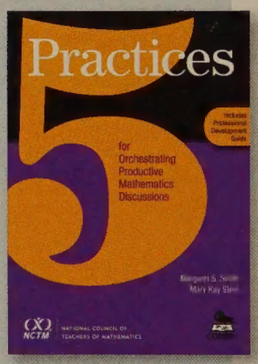

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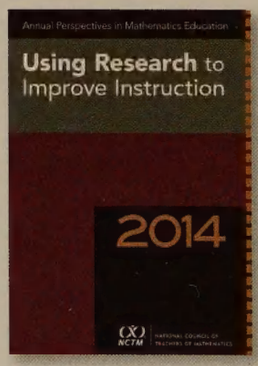

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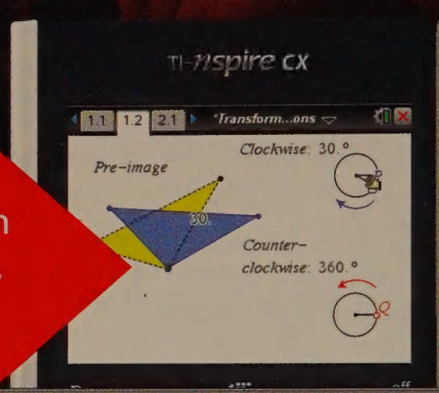

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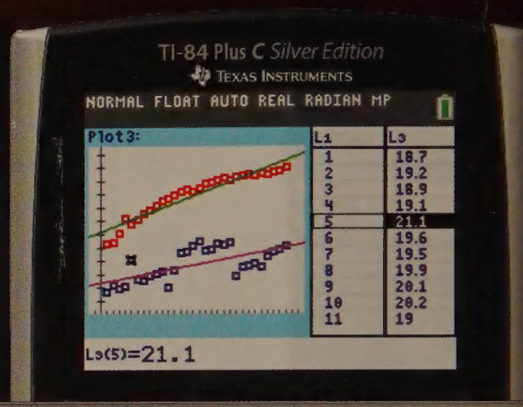

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